Science Arena Publications
Specialty Journal of Engineering and Applied Science
ISSN: 2520-5943
Available online at www.sciarena.com
2019, Vol, 4 (4): 31-40

Bermudan Transaction Option Pricing Under Heston Model

Saeedeh Khaziravi

Department of Mathematics, Faculty of Computer and Mathematics, University of Khansar, Khansar, Iran.

Abstract: *In the present study, an effective numerical method is offered based on the Fourier series for pricing the Bermudan transaction option and discrete observation option under Heston stochastic volatility model. In fact, it is through introducing Heston Model and by the assistance of Fourier series that the 2D pricing formulas of Bermudan transaction option and barrier discrete options are presented. To do so, numerical integration regulations are used and the error convergence is investigated through performing numerical tests.*

Keywords: *Heston model, Bermudan transaction option, Fourier Cosine expansion, numerical integration rules.*

INTRODUCTION

The models including the stochastic volatility have been widely used for modeling the transaction options in the markets. Amongst these, Heston stochastic volatility models can be pointed out that has been cited in (Heston, 1993). In these models, the stock price logarithm is modeled by means of the square root process.
European option pricing has been particularly more frequently used in contrast to the other transaction options. Using the cosine method and by the assistance of Fourier series (Lord et al., 2008; Lord et al., 2009), the European option pricing formula and 1D Bermudan option pricing formula can be obtained. To do so, the FFT algorithm is applied to enhance the speed and efficiency of the method. The integration methods used in this method, as well, are based on the Fourier series and need the possession of a characteristic function (Fourier transform of the base stock price's density probability function).
The specified function of Heston model has been obtained in (Heston, 1993). Next, the cosine method presented in (Lord et al., 2008; Lord et al., 2009) will be utilized for Bermudan transaction option pricing in a 2D state as well as for discrete observations of barrier transaction option in a general state. The following challenges are encountered for doing so by the assistance of the Heston Model:

1) Nearly unique behavior of the variance probability density: the variance is exhibited in Heston model by non-central Chi-square distribution and the variance density severely grows on the left tail (sequence) side per some of the corresponding parameters, i.e. the density values are inclined towards infinity when variance tends towards zero and the integral limit cut causes a large deal of truncation error.
2) Integration core that is not vividly clear: for path-dependent transaction options, the pricing formula needs 2D integration on the stock price and variance logarithm. The joint probability distribution

density function does not have a closed-form answer, i.e. it is not clear; so, its characteristic function is utilized for determining it.

3) The complexity of the numerical integration calculations: the direct use of the numerical integration rules for the transaction option through premature exertion under the Heston model would lead to the complexity of the numerical integration calculations that takes a considerable amount of the processor's time.

Heston Model:

The stochastic Heston equations are defined in the form of stock price logarithm dynamics, x_t, and variance, V_t, as shown below:

$$dx_t = \left(\mu - \frac{1}{2}v_t\right)dt + \rho\sqrt{v_t}dw_{1,t} + \sqrt{1-\rho^2}\sqrt{v_t}dw_{2,t} \qquad (1)$$

$$dv_t = \lambda(\overline{v} - v_t)dt + \eta\sqrt{v_t}dw_{1,t} \qquad (2)$$

Where, λ, V and η are three parameters respectively designating the mean reciprocation speed, variance level and the turbulence of the volatility process. $\omega_{1,t}$ and $\omega_{2,t}$ are two independent Brownian motions and ρ is the correlation between the logarithm of the stock price and variance process. The process $\sqrt{v_t}$ existent in (Lord et al., 2008) prevents the V_t values from becoming negative. Feller Condition $2\lambda\nu \geq \eta^2$ guarantees that V_t remains positive; it means that the variance process will be strictly positive if the Feller condition holds otherwise, i.e. if the Feller condition does not hold, the variance process reaches zero border which is intensively reflexive. Here, it is assumed that the Feller condition holds.

Using the relationship:

$$p(v_t|v_s) = \xi e^{-u-v}\left(\frac{-v}{u}\right)^{\frac{q}{2}}.I_q\left[2(uv)^{\frac{1}{2}}\right]$$

From (Cox, Ingersoll and Roosm, 1985) and the definitions:

$$q := \frac{2\lambda\overline{v}}{\eta^2} - 1 \qquad ,\xi = \frac{2\lambda}{(1-e^{-\lambda(t-s)})\eta^2}$$

For density function, v_t, on the condition that $v_s$is available at time s, the following relationship holds:

$$(v_t|v_s) = \xi e^{-\xi v_s e^{-\lambda(t-s)} - \xi v_t}\left(\frac{\xi v_t}{\xi v_s e^{-\lambda(t-s)}}\right)^{\frac{q}{2}} I_q\left[2(\xi v_s e^{-\lambda(t-s)}\xi v_t)^{\frac{1}{2}}\right]$$

Next, $(-\xi)$ is factored from the first sentence as a factorial agent and the following relationship is obtained with the omission of ξ from the second term and using the radical display of the term $I_q(.)$:

$$(v_t|v_s) = \xi e^{-\xi(v_s e^{-\lambda(t-s)} + v_t)}\left(\frac{v_t}{v_s e^{-\lambda(t-s)}}\right)^{\frac{q}{2}} I_q\left[2(\xi^2 v_s v_t e^{-\lambda(t-s)})^{\frac{1}{2}}\right]$$

Resultantly:

$$(v_t|v_s) = \xi e^{-\xi(v_s e^{-\lambda(t-s)} + v_t)}\left(\frac{v_t}{v_s e^{-\lambda(t-s)}}\right)^{\frac{q}{2}} I_q\left[2\xi\, e^{-\lambda/2(t-s)}(v_s v_t)^{\frac{1}{2}}\right] \qquad (3)$$

Where, I_q is the first kind modified Bessel function of the rank q and z is defined in the following form therein:

$$z = 2\xi\ e^{-\lambda/2(t-s)}(v_s v_t)^{\frac{1}{2}}$$

Feller condition is equivalent to $q \geq 0$ because:

$$q \geq 0 \quad ,q := \frac{2\lambda\overline{v}}{\eta^2} - 1 \ \geq 0 \rightarrow \ \frac{2\lambda\overline{v}}{\eta^2} \geq 1 \ \rightarrow \ 2\lambda\overline{v} \ \geq \eta^2$$

Such a behavior on the left tail side might bring about an increase in error, especially for the option pricing methods based on integration in which their integration limits need cutting.

Brody and Kaya offered their exact simulation method in (Broadie and O. Kaya, 2006) for finding the stock price logarithm paths. The following relationships can be obtained by the integration of the relations (3-1) and (3-2):

$$x_t - x_s = \mu(t-s) - \frac{1}{2}\int_s^t v_\tau d\tau + \rho\int_s^t \sqrt{v_\tau} d\omega_{1,\tau} + \sqrt{1-\rho^2}\int_s^t \sqrt{v_\tau} d\omega_{2,\tau} \quad (4)$$

$$x_t - x_s = \lambda\overline{v}(t-s) - \lambda\int_s^t v_\tau d\tau + \eta\int_s^t \sqrt{v_t} d\omega_{1,\tau} \quad (5)$$

Equation (5) can be rewritten in the form of an equation for $\int_s^t \sqrt{v_t}\, d\omega_{1,T}$. An exact formula is obtained for x_t as shown below through inserting it in relation (4):

$$x_t - x_s = \lambda\overline{v}(t-s) - \lambda\int_s^t v_\tau d\tau + \eta\int_s^t \sqrt{v_t} d\omega_{1,\tau} \Rightarrow$$

$$\int_s^t \sqrt{v_t} d\omega_{1,\tau} = \frac{1}{\eta}\left[v_t - v_s - \lambda\overline{v}(t-s) + \lambda\int_s^t v - \tau d\tau\right]$$

It can be concluded by embedding the above relation in (5) that:

$$x_t - x_s = \mu(t-s) - \frac{1}{2}\int_s^t v_t d\tau + \frac{\rho}{\eta}\left[v_t - v_s - \lambda\overline{v}(t-s) + \lambda\int_s^t v - \tau d\tau\right] + \lambda\frac{\rho}{\eta}\int_s^t v_\tau d\tau \Rightarrow$$

$$x_t - x_s = \mu(t-s) + \frac{\rho}{\eta}v_t - v_s - \lambda\overline{v}(t-s)) + (\lambda\frac{\rho}{\eta} - \frac{1}{2})\int_s^t v_\tau d\tau + \sqrt{1-\rho^2}\int_s^t \sqrt{v_\tau} d\omega_{2,\tau} \quad (6)$$

Equation (6) can be used for a sample of x_t in a state that the amounts of v_t and $\int_s^t v_t d_T$ variance are clear, a sample of v_t can be approximated by non-central chi square distribution. Moreover, a sample of $\int_s^t v_t\, d_T$can be obtained through the retrieval of the characteristic function the closed form of which takes the following form:

$$\Phi(v, v_t, v_s) := E\left[e^{iv\int_s^t v_\tau d\tau} | v_t, v_s\right]$$

$$= \frac{I_q\left[\sqrt{v_t v_s}\frac{4\gamma(v)e^{-\frac{1}{2}\gamma(v)(t-s)}}{\eta^2(1-e^{-\gamma(v)(t-s)})}\right]}{I_q\left[\sqrt{v_t v_s}\frac{4\gamma e^{-\frac{1}{2}\gamma(t-s)}}{\eta^2(1-e^{-\gamma(t-s)})}\right]} \times \frac{\gamma(v)e^{-\frac{1}{2}(\gamma(v)-\lambda)(t-s)}(1-e^{-\lambda(t-s)})}{\lambda(1-e^{-\lambda(t-s)})}$$

$$\times \exp\left(\frac{v_t+v_s}{\eta^2}\left[\frac{\lambda(1+e^{-\lambda(t-s)})}{(1-e^{-\lambda(t-s)})} - \frac{\gamma(v)(1+e^{-\gamma(v)(t-s)})}{\lambda(1-e^{-\gamma(v)(t-s)})}\right]\right. \quad (7)$$

Where, $q = \frac{2\lambda v}{\eta^2}$ and $I_q(x)$are the q-order modified Bessel function of the type one and $\gamma(v)$is the variable defined as demonstrated below:

$$\gamma(v) = \sqrt{\lambda^2 - 2i\eta^2 v} \qquad (8)$$

Pricing Method for Bermudan Options:

In this section, a formula is obtained for Bermudan options' pricing under Heston model that includes a dual integral one part of which has the exact answer. In the calculation of this dual integral, a discrete formula is used for expanding the Fourier's cosine series to calculate the integral part of the core that is not clear in closed form and the numerical integration is applied for the integral of part of the core that is certain. On the other hand, an efficient and effective algorithm is introduced for calculating the discrete formula by the assistance of the FFT algorithm.

1. **Transactions' Pricing:**

   For a European option specified for a term s and matured at t wherein $0 < s < t$, the risk neutral pricing formula takes the following form:

$$v(x_s, \sigma_s, s) = e^{-r(t-s)} E_s^Q [v(x_s, \sigma_s, s)] \qquad (9)$$

Where, $v(x_s, \sigma_s, s)$ is the option price at time s and r is the risk-free interest rate and E_S^Q is the hope operator under the size of the neutral risk Q provided that it contains the information at time s. Markov property helps the attainment of a Bermudan option between two premature exertion dates, (16), using the neutral risk valuation formula. The arbitrage-free Bermudan option at any premature date would be equal to the maximum continuous amount and the option exertion gain. For M premature exertion time and $\Delta t := t_{m+1} - t_m \; and \; t_M \equiv T \; with \; \tau := \{t_m, t_m < t_{m+1} \,|m = 0,1,2, \ldots, M\}$, Bermudan option pricing formula takes the following form:

$$v(x_{t_m}, \sigma_m, t_m) = \begin{cases} g(x_{t_m}, t_m) & for \quad m = M \\ \max\,[c(x_{t_m}, \sigma_m, t_m), g(x_{t_m}, t_m)] & for\; m = 1,2, \ldots. M-1 \\ c(x_{t_m}, \sigma_m, t_m) & for \;\; m = 0 \end{cases} \qquad (10)$$

Where, $q(x_\tau, \tau)$is the gain function at the time τ and $c(x_\tau, \sigma_\tau, \tau)$ is the continuous amount at the time τ. Next, xm and $\sigma_m$symbols are used instead of x_{t_m} and σ_{t_m} for simplicity. The continuous amount is also shown in the following form:

$$c(x_{t_m}, \sigma_m, t_m) = e^{-r\Delta t} E_{,t_m}^Q [v(x_{m+1}, \sigma_{m+1}, t_{m+1})] \qquad (11)$$

So, it can be written that:

$$c(x_{t_m}, \sigma_m, t_m) = e^{-r\Delta t}. \int_R \int_R v(x_{m+1}, \sigma_{m+1}, t_{m+1}) \rho_{x|\ln(v)} (v)(x_{m+1}, \sigma_{m+1} | x_m, \sigma_m) d\sigma_{m+1} dx_{m+1} \qquad (12)$$

It can also be written that:

$$c(x_{t_m}, \sigma_m, t_m) = e^{-r\Delta t}. \int_R \left[\int_R v(x_{m+1}, \sigma_{m+1}, t_{m+1}) \rho_{x|\ln(v)}(x_{m+1} | \sigma_{m+1} x_m, \sigma_m) \; dx_{m+1}\right] . pln(v)(\sigma_{m+1}|\sigma_m) d\sigma_{m+1} \qquad (13)$$

Next, the numerical solution of the problem is dealt with based on the equations (10) and (13). The internal integral in the above relation is the very pricing formula for the European option defined between t_m and t_{m+1} that specifies the variance amount in a future time. The scaled asset's logarithm price will be more repeatedly used in the forthcoming sections than before and it is defined in the following form:

$$x_m = \ln\left(\frac{S_m}{K}\right)$$

2. The Density Retrieved based on the Fourier Cosine Expansion:

The essential idea of the cosine method, as stated in (Lord et al., 2008), is the approximation of the base density function that is most often a smooth function giving real quantities. It has to be noted that the Fourier series coefficients have direct relationships with the characteristic function. Next, a cross-section is defined for span $[a,b] \subset \mathbb{R}$ in such a way that it also holds for the following relation, i.e.

$$\int_a^b \rho_{x|\ln(v)}(v)(x_{m+1}, \sigma_{m+1}|x_m, \sigma_m) \leq TOL_x \quad (14)$$

Where, TOLx is the error threshold. The same way that the following interval has been defined in (Lord et al., 2008) as a cross-section for span $[a,b] \subset \mathbb{R}$of Heston Model, the span is also used herein.

$$[a,b] := [\xi_1 - 12\sqrt{|\xi_2|}, \xi_1 + 1212\sqrt{|\xi_2|} \quad (15)$$

Where, $\xi_n$denotes the nth indicator of the stock price logarithm process. With the integral interval [a, b] that also holds in the above relation, the goal is retrieving the density function by the assistance of Fourier Cosine Series Expansion. Assume that:

$$\rho_{x|\ln(v)}\left(x_{m+1}|\sigma_{m+1}, x_m, \sigma_m\right) = \sum_{n=0}^{f\infty} \rho_n(\left(\sigma_{m+1}, x_m, \sigma_m\right)\cos\left(n\pi \frac{x_{m+1}-a}{b-a}\right) \quad (16)$$

Where, $\sum'$ shows that the first sigma member is multiplied by ½. ρ_n is a Fourier cosine coefficient that is obtained in the following form, i.e.

$$\rho_n\left(\sigma_{m+1}, x_m, \sigma_m\right) := \frac{2}{b-a}\int_a^b \rho_{x|\ln(v)}(v)(x_{m+1}, \sigma_{m+1}|x_m, \sigma_m) \cos\left(k\pi \frac{x_{m+1}-a}{b-a}\right)$$

According to the fact that there is a direct relationship between the coefficients and ChF (characteristic function) and, on the other hand, it was stated that there is no exact answer for $\rho_{x|In(v)}$, the characteristic function defined for the function $\rho_{x|In(v)}$ is used. The term cos(.) can be written in the following form:

$$\text{Real}[\exp\left(\text{i}n\pi \frac{x_{m+1}-a}{b-a}\right)] = \text{Real}[\exp\left(in\pi \frac{x_{m+1}}{b-a}\right)\exp\left(-in\pi \frac{a}{b-a}\right)] = Real[\varphi\left(\frac{n\pi}{b-a}\right)\exp\left(-in\pi \frac{a}{b-a}\right)]$$

$$\rho_n\left(\sigma_{m+1}, x_m, \sigma_m\right) \approx \frac{2}{b-a} Real\left\{\varphi\left(\frac{n\pi}{b-a}; x_m, \sigma_{m+1}, \sigma_m\right) exp\left(-in\pi \frac{a}{b-a}\right)\right\} \quad (17)$$

Where, $\varphi(\theta; x, \sigma_{m+1}, \sigma_m)$has been obtained. It has to be noted that when the span [a, b] is expanded, p_n is approximated with a machine precision (device needed for approximation). Next, the sum of the series (sigma) is cut in the relations (3) and (22).

Replacing from relation (16) in relation (17) and cutting the series with N terms, the following formula can be obtained which is an appropriate approximation for the intended density function, i.e.

$$\rho_{x|\ln(v)}\left(x_{m+1}|\sigma_{m+1}, x_m, \sigma_m\right) =$$

$$\sum_{n=0}^{N-1} \frac{2}{b-a} Real\left\{\varphi\left(\frac{n\pi}{b-a}; 0, \sigma_{m+1}, \sigma_m\right) exp\left(in\pi \frac{x_m - a}{b-a}\right)\right\} \cos\left(n\pi \frac{x_{m+1}-a}{b-a}\right) + \epsilon_{cos} \quad (18)$$

Where, the function
$\varphi\left(\omega; x_m, \sigma_{m+1}, \sigma_m\right) = \exp\left(i\omega x_m\right)\varphi(\omega; 0, \sigma_{m+1}, \sigma_m)$ has been used and Xm can be separated from the σ–dependent terms and be shown like a simple term. This is very much useful in Bermudan calculations. Based on the theorem, the justification of which has been given in (Lord et al., 2009), the error of this approximation, i.e. ε_{cos}, exponentially decreases with respect to N provided that the cut section is sufficiently large.

3. **Pricing Formula based on Discrete Fourier Transform:**
Equation (10) wherein a formula was constructed for Bermudan option pricing shows that the option price is a continuous quantity at the time t_0. As it is shown in relation (13), its amount depends on the continuous quantity at times $t_1, t_2, \ldots t_M$. The option price at t_0 can be retrieved through retrogressive reciprocation in time as shown in (Lord et al., 2009).

Numerical Results:

The convergence of the obtained error is analyzed assuming $l = a$; $u = b$ by means of discrete pricing of the barrier transactions' option. To compute the exact values in a high precision (to the eight decimal), use is made of the European option pricing mentioned in (Lord et al., 2008). According to (Andersen, 2008), three types of tests are carried out one of which belongs to q>0 and the other two are pertinent to $q \in [-1,0]$.

Test Number One: $\eta = 0.5$, $\lambda = 5$, $\overline{v} = 0.04$, $T = 1$: $(q = -0.84)$
Test Number Two: $\eta = 0.5$, $\lambda = 0.5$, $\overline{v} = 0.04$, $T = 1$: $(q = -0.84)$
Test Number Three: $\eta = 1$, $\lambda = 0.5$, $\overline{v} = 0.04$, $T = 10$: $(q = -0.96)$

The computer used in the present study is a standard laptop with an R input and 2.2GHz processor and a 4-Gb memory. The numerical methods for premature exertion or exercising of barrier transaction options are usually based on the finite difference for PDEs (Ito and Toivanen, 2009) or tree methods (Vellekoop and Nieuwenhuis, 2009). The tree-form results and the finite differences for this set of parameters that have been introduced in the abovementioend tests are yet to be published. The other parameters that are used for showing sales per α=-1, include the following quantities:

$$\rho = -0.09, \quad v_0 = 0.04, \qquad S_0 = 100, \qquad K = 100, \qquad r = 0$$

Next, error convergence in J is investigated for Heston pricing models based on the numerical integration rule of Fauss Legendre.
Then, a prespecified TOL truncation error threshold is determined per 10^{-4}, 10^{-6} and 10^{-8} values. In between, the number of the observed dates is set at 12 and N is assumed to be 2^7.
Table (1) shows that the error rate is found being decreased per every amount of J when having $TOL = 10^{-4}$ and that although the values change about a number, they show a descending trend per every $TOL \leq 10^{-6}$. In the end, when $TOL \leq 10^{-8}$, the error values start decreasing at a higher speed. Consequently, the error is still convergent with the increase in J and the reduction in TOL. The results obtained for q>0 are within the limit of a fraction of a second hence featuring a high accuracy. The exact amount for the European option is 7.5789038982 as shown in Table (1).

Table 1: Fourier Cosine Series Transform and Gauss Legendre Rule

(J=2^d^)	TOL= 10^{-2}		TOL= 10^{-6}		TOL= 10^{-8}	
d	Frror	Time (sec)	Frror	Time (sec)	Frror	Time (sec)
4	$-7.51*10^{-2}$	0.12	$1.02*10^{-2}$	0.12	1.141	0.12
5	$-3.95*10^{-2}$	0.43	$-1.85*10^{-5}$	0.42	$2.99*10^{-5}$	0.40
6	$-3.95*10^{-2}$	1.69	$-1.54*10^{-5}$	1.59	$-6.41*10^{-6}$	1.54
7	$-3.95*10^{-2}$	6.88	$-1.34*10^{-5}$	7.07	$-6.32*10^{-7}$	6.49

Table 2: Fourier Cosine Series Transform and Gauss Legendre Rule per the negative q values

	(q = -0.84) test nember2				(q = -0.96) test nember3			
($J=2^d$)		Time (sec)				Time (sec)		
d	Frror	Main	Input	Total	Frror	Main	Input	Total
6	5.63	0.18	2.85	3.03	-22.7	0.18	2.93	3.11
7	$6.89*10^{-3}$	0.18	2.85	13. 3	$-8.51*10^{-2}$	0.53	11.55	12. 1
8	$-2.12*10^{-5}$	4.07	52.32	56.4	$-1.60*10^{-3}$	4.00	51.74	55.7

Now, the turn comes for the more critical cases of the test wherein $q \to -1$ is investigated. For the set of the given parameters, the combined trapezoidal axiom (also the combined Simpson rule) is not satisfactory because it needs very large quantities of J to reach the desired precision.

But, the Gauss Legendre rule can give optimal results for small J values. The results obtained from Gauss Legendre rule have been numerically shown in Table (2) for variance logarithm aspect wherein $TOL = 10^{-7}$ *and* $M = 12$ *and* $N = 10^{28}$and the exact values for the European option in the second and the third tests are 6.2710582179 and 13.0842710701, respectively.

In comparison to the test number one, the real error of the second and the third tests were found being rapidly decreasing for the same values of J and N. however, the error convergence in J is still very fast. Also, the results of Table (2) show that a long calculation time is consumed when $q \to -1$, the initial value assignment stage; so, it can be stated that it is due to the existence of Bessel Function that a large amount of time is consumed for calculation. On the other hand, the calculation time of the primary loop includes less than 8% of the total time.

Table 3: error convergence per the ascending values of M

Test	M		
	40	20	10
test nember1 (q = 0.6)	$-4.92*10^{-6}$	$-3.13*10^{-6}$	$-2.14*10^{-6}$
test nember2 (q = -0.84)	$-7. 02*10^{-4}$	$-2.71*10^{-5}$	$-2.56*10^{-5}$

Table 4: convergence of Bermudan transaction option to the American option per every correlation coefficient and fixed q

S_0	8	9	10	11	Time		
10_M	2.000000	1.107621	0.520030	0.213677	Total	Input	Main ring
M=10	$-1.18*10^{-2}$	$-4.79*10^{-3}$	$-2.85*10^{-3}$	$-1.31*10^{-3}$	6.9	6.34	0.57
M=20	$-9.54*10^{-3}$	$-2.39*10^{-3}$	$-1.40*10^{-3}$	$-6.65*10^{-3}$	7.5	6.36	1.13
M=40	$-5.14*10^{-3}$	$-1.07*10^{-3}$	$-5.50*10^{-4}$	$-2.54*10^{-4}$	8.9	6.57	2.32
M=80	$-2.83*10^{-3}$	$-2.86*10^{-4}$	$-2.75*10^{-5}$	$-5.42*10^{-5}$	14.1	7.35	6.70

Error propagation is investigated per time. To do so, N and J are presumed to be fixed and the error convergence is measured for the incremental values of M that have been given in Table (3). The results confirm that the local error grows very slowly for q>0 and it somehow grows rapidly for $q \in [-1,0]$; the general error is reduced with assigning larger quantities to J and/or N and doubling of Parameter M causes doubling of CPU time in the primary loop and the convergence is still established in this state.

Table 5: convergence of Bermudan transaction option to the American option with a negative q value and correlation coefficient

	S_0			Time		
M	90	100	110	Total	Input	Main ring
20	0.9783714	3.204734	0.9273568	68.9	58.2	10.7
40	9.9916484	3.2073345	0.9281068	81.9	59.3	22.6
60	9.9957789	3.2079202	0.9280425	93.2	59.4	33.8

Bermudan Option Pricing Algorithm:

Step One:
Using the relation $\ln(E(v_t)) = \ln[v_0\, e^{-\lambda T} + \bar{v}(1-e^{-\lambda T})]$ and assuming $T = 1, v_0 = 0.04, \bar{v} = 0.04, \lambda = 5$ values for the first test and $T = 1, v_0 = 0.04, \bar{v} = 0.04, \lambda = 0.5$values for the second and the third tests, three values are obtained for $ln(E(v_t))$per every test; embedding of each value in the relation $[a_v^0, b_v^0] = [\ \ln E(v_t) - \frac{5}{1+q}, \ln E(v_t) + \frac{2}{1+q}]$, and setting q equal to 0.6 for the first test, equal to -0.84 for the second test and equal to -0.96 for the third test gives the [a, b] span.

Step Two:
Assuming the sale state for Bermudan Option, its gain function can be written in the form of $g(y) = [\alpha k(e^y - 1)]^+$ and $G_n(l,u) = \frac{2}{b-a}\int_l^u k(1-e^y)\cos(n\pi\frac{y-a}{b-a})dy$; having the values *l=a* , *u=b* and setting k equal to 100, the amount for sale state would be the answer to the integral $V_{n.j}(t_M) = G_n(l,u) = \frac{2}{b-a}\int_l^u k(1-e^y)\cos(n\pi\frac{y-a}{b-a})dy$ that would take a different value per every k. It has to be pointed out that α=-1 for sale state.

Step Three:
Matrix $\bar{\varphi}(\xi_j)$is defined according to the relation $\bar{\varphi}\left(\frac{n\pi}{b-a}, \xi_j, \xi_p\right) := \rho_{\ln(v)}(\xi_j|\xi_p).\varphi(\frac{n\pi}{b-a}, 0, e^{\xi_j}, e^{\xi_p})$ wherein *p=0,1, …, J-1* and *j=0,1,…,J-1* . Next, $\rho\, ln(v)$in relation (9-3) and the term $\varphi(0)$ in the relation (14-3) would be obtained with various quantities for every test. In fact, the equations, $q = \frac{2\lambda\bar{v}}{\eta^2} - 1$ and $\xi = \frac{2\lambda}{(1-e^{-\lambda(t-s)})}\eta^2$ can be used when having $\bar{v}, \lambda, q$ to obtain the values of η and ξ.

Primary Loop:

Step One:
The premature exertion points of Bermudan transaction option can be obtained by solving the equation $c_3(y, \xi_p, t_m) - g(y) = 0$ based on the Newton method. To do so, by obtaining the function's derivative and having a point as the initial assumption and using the formula $x_{n+1} = x_n - \frac{f(x)}{\acute{f}(x)}$, the equation roots that are the premature exertion spots can be attained in the form of $y = x^*(\xi_p, t_{M-1})$.

Step Two:

The first line and column of the Ms and Mc matrixes allows the attainment of Henkel and Horseshoe Matrices.

Step Three:
Calculation of $\widehat{\beta}_j(t_m) = \left[\widehat{v}(t_m).\tilde{\varphi}\left(\xi_j\right)\right]W$ per every $\tilde{\varphi}(\xi_j).j = 0,1,\dots,J-1$ was conducted in the third step of the initial value assignment and $\hat{v}(t_m)$ is obtained for the sale state from the relation (43-3).

Step Four:
The first member of $\hat{\beta}_j(t_m)$ that was obtained in the previous step is multiplied by ½ and $\hat{\beta}_j^{'}(t_m)$ is obtained.

Step Five:
The vectors $e^{-r\Delta t}Re\left\{(M_s + M_c)\widehat{\beta_j^{'}}(t_{m-1})\right\}$ *and* $\hat{C}(t_m)$are obtained using FFT algorithm. In this step, matrices Mc and Ms exist and $\hat{\beta}_j^{'}(t_{m-1})$ is obtained using the previous step.

Step Six:
Rewriting of $V^{'}(t_m)$
The Last Step:
Calculation of $\hat{v}(\chi, \xi_j, t_0)$. Spline interpolation is used for $\hat{v}(\chi, \sigma_0, t_0)$.

4. **The Numerical Results for Bermudan Option:**
The most frequently used parameters for the American option under Heston dynamics through the test number four is offered in the following form:
Test Number 4 (q=0.98):

$$S_0 = \{8,9,10,11,12\} \quad K = 10 \quad T = 0.25 \quad r = 0.1 \quad \lambda = 5 \quad \eta = 0.9 \quad \overline{v} = 0.0625 \quad \rho = 0.1$$

Where, q>0. It is expected that the pricing performance is very precise and effective. The results have been given in table (4) wherein the system processor or CPU time has been calculated and recorded for five different values of S_0. On the other hand, $TOL = 10^{-7}$ *and* $N = J = 2^7$ that has been obtained using Gauss Legendre rule and cosine calculations for q=0.098. The results of the Bermudan sale option per every negative $q \in [-1,0]$ and P have been given in table (5). They have not been computed before for negative ρ values.
Test Number Five (q=-0.47):

$$S_0 = \{90,100,110\} \quad K = 100 \quad T = 0.25 \quad r = 0.04 \quad \lambda = 1.15 \quad \eta = 0.39 \quad \overline{v} = 0.0348 \quad v_0 = 0.0348$$

Where, $TOL = 10^{-7}$ *and* $N = J = 2^7$ *and* $q = -0.47$. According to the values given in the two tables, the convergence of Bermudan Option to the exact value of American option mentioned in (Vellekoop and Nieuwenhuis, 2009) has been shown.

Conclusion:

The present study offered a strong and effective method for Bermudan pricing and discrete observations of the barrier option under the Heston stochastic volatility based on Fourier Series. The problem related to the quasi-unique behavior in the left side tail of Heston variance density was solved by changing a variable in the variance logarithm domain. A discrete pricing formula was obtained based on cosine series expansion in the logarithm dimension of the stock and a numerical integration rule in the variance logarithm aspect. The fast error convergence was shown in a discussion on the error analysis and determination of some of the parameters using the numerical integration methods for pricing method. Although some of the parameters did not hold for Feller condition in premature option pricing, the offered method gave the option's prices with high accuracy for a fraction of second. On the other hand, in the stage of setting the initial values of the algorithm, although most of the calculation time was consumed for Bessel Function calculation, error convergence was numerically observed. The method presented herein can be applied like a resilient rule for stochastic volatility methods like Heston model which has stochastic interest rates; of course, this works when the characteristic function or the joint probability distribution density function of the status variables are clear.

Reference

1. J. C. Cox, J. E. Ingersoll, and S. A. Roos, "A theory of the term structure of interest rates", Econometrica, 53 (1985), 385-407.
2. K. Ito and J.Toivanen, "Lagrange multiplier approach with optimized finite difference stencils for pricing american options under stochastic volatility ", SIAM. J. Sci. Comput. 31 (2009) 2646-2664.
3. L. Andersen, "Simple and efficient simulation of the Heston stochastic volatility model", J. Comput Finance, 11 (2008), 1-42.
4. M. Broadie and O. Kaya , "Exact simulation of stochastic volatility and other affine jump diffusion processes", Oper. Res. 54 (2006), 217-231.
5. M. Vellekoop and H. Nieuwenhuis, "A tree-based mehod to price american option in the Heston model", J. Comput. Finance, 13 (2009) 1-21.
6. R. Lord, F. Fang, F. Bervoets, and C. W. Oosterlee, "A novel pricing method for European options based on Fourier- cosine series expansions", SIAM. J. Sci. Comput. 31 (2008), 826-848.
7. R. Lord, F. Fang, F. Bervoets, and C. W. Oosterlee, "Pricing early-exercise and discrete barrier options by Fourier -cosine series expansions", Number. Math. 114 (2009), 27-62.
8. S. Heston, "A closed form solution for options with stochastic volatility with applications to bond and currency options", Rev. Finance. stud. 6 (1993) 327- 343.