



LATTICE STRUCTURES OF DOUBLE-FRAMED FUZZY SOFT LATTICES

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Abstract: *In this paper, we introduce the notion of double-framed fuzzy quotient lattice in a fuzzy lattice and then some basic properties are investigated. Characterizations of double-framed fuzzy quotient lattices are given. Using a collection of lattices, a double framed fuzzy quotient lattice is established. The notion of fuzzy quotient lattice relation on the family of all I double-framed fuzzy sub lattices of L are discussed upper and lower level sets of fuzzy quotient lattices are studied.*

Keywords: *d Fuzzy set, Fuzzy lattice, Fuzzy quotient Lattice, level cut, double-framed fuzzy quotient sub lattice, Homomorphism*

INTRODUCTION

The theory of fuzzy sets proposed by (Zadeh, 1965) in 1965, has achieved a great success in various fields. In 1999, Molodtsov (Molodtsov, 1999) introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainties. This so-called soft set theory is free from the difficulties affecting existing methods. Presently, works on soft set theory are progressing rapidly. (Maji et al. 2003) defined several operations on soft sets and made a theoretical study on the theory of soft sets. (Atagn and Sezgin, 2011) defined the concepts of soft sub rings and ideals of a ring, soft subfields of a field and soft sub modules of a module and studied their related properties with respect to soft set operations. Also union soft sub structures of near-rings and near-ring modules are studied in (Maji, 2003). (Hayat et al. 2016) defined applications of double-framed soft ideals in BEalgebra. (Jun et. al, 2012, 2013) introduced the notion of double-framed soft sets (briefly, DFSsets), and applied it to BCK/BCI- algebras. They discussed double-framed soft algebras (briefly, DFS-algebras) and investigated related properties. (Hadipour, 2014) defined double-framed soft BF-algebras and (Yongukcho et al. 2015) studied on double-framed soft Near-rings. Cagman et al. defined two new types of group actions on a soft set, called group SI-action and group SU-action (Cagman, 2012), which are based on the inclusion relation and the intersection of sets and union of sets, respectively. (Ali et al. 2009) introduced several operations of soft sets. (Yongukcho et al. 2015) studied on double-framed soft Near-rings derive some properties. (Saibaba, 2008) initiated the study of L -fuzzy lattice ordered groups

and introduced the notion L fuzzy subl groups. (Goguen, 1967) replaced the valuation set $[0,1]$ by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L -fuzzy sets. K.V. Thomas and Latha. S. Nair studied Rough intuitionistic fuzzy sets in a lattice (Thomas et al, 2011). In this paper, we investigate the notion of double-framed fuzzy quotient lattice in a fuzzy lattice and then some basic properties are investigated. Characterizations of double-framed fuzzy quotient lattices are given. Using a collection of lattices, a double-framed fuzzy quotient lattice is established. The notion of fuzzy quotient lattice relation on the family of all double-framed fuzzy sub lattices of L are discussed upper and lower level sets of fuzzy quotient lattices are studied.

1 Preliminaries:

In this section, we recall basic definitions of soft set theory that are useful for subsequent sections. For more detail see the papers (Molodtsov, 1999. Molodtsov,2004)

Throughout the paper, U refers to an initial universe, E is a set of parameters and $P(U)$ is the power set of U . \subset and \supset stand for proper subset and super set, respectively.

Definition 2.1 (9). For any subset A of E , a soft set λ_A over U is a set, defined by a function λ_A , representing the mapping $\lambda_A: E \rightarrow P(U)$. A soft set over U can also be represented by the set of ordered pairs $\lambda_A = \{(x, \lambda_A(x)); x \in E, \lambda_A(x) \in P(U)\}$. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.2 (9). Let $\lambda, \mu \in S(U)$. Then

- (i) If $\lambda(e) = \emptyset$ for all $e \in E$, λ is said to be a null soft set, denoted by \emptyset .
- (ii) If $\lambda(e) = U$ for all $e \in E$, λ is said to be an absolute soft set, denoted by U . (iii) λ is a soft subset of μ , denoted $\lambda \subseteq \mu$, if $\lambda(e) \subseteq \mu(e)$ for all $e \in E$.
- (iv) Soft union of λ and μ , denoted by $\lambda \cup \mu$, is a soft set over U and defined by $\lambda \cup \mu: E \rightarrow P(U)$ such that $(\lambda \cup \mu)(e) = \lambda(e) \cup \mu(e)$ for all $e \in E$.
- (v) $\lambda = \mu$, if $\lambda \subseteq \mu$ and $\lambda \supseteq \mu$.
- (vi) Soft intersection of λ and μ , denoted by $\lambda \cap \mu$, is a soft set over U and defined by $\lambda \cap \mu: E \rightarrow P(U)$ such that $(\lambda \cap \mu)(e) = \lambda(e) \cap \mu(e)$ for all $e \in E$.
- (vii) Soft complement of λ is denoted by λ^c and defined by $\lambda^c: E \rightarrow P(U)$ such that $\lambda^c(e) = \cup/\lambda(e)$ for all $e \in E$.

Definition 2.3. Let E be a parameter set, $S \subset E$ and $\lambda: S \rightarrow E$ be an injection function.

Then $S \cup \lambda(s)$ is called extended parameter set of S and denoted by ξ_s . If $S = E$, then extended parameter set of S will be denoted by ξ .

Definition 2.4 (7). A double-framed pair $(\bar{(\cdot)}, \underline{(\cdot)}) : G$ is called a double-framed soft set (briefly DFS-set) over U where $\bar{(\cdot)}$ and $\underline{(\cdot)}$ mapping from A to $P(U)$.

For a DFS-set $(\bar{(\cdot)}, \underline{(\cdot)}) : G$ over U and two subsets γ and δ of U , the γ -inclusive set and the δ -exclusive set of $(\bar{(\cdot)}, \underline{(\cdot)}) : G$ denoted by $i_A(\bar{(\cdot)}, \gamma)$ and $e_A(\underline{(\cdot)}, \delta)$ respectively, are defined as follows.

$i_A(\bar{(\cdot)}, \gamma) = \{x \in A/\gamma \subseteq \bar{(\cdot)}(x)\}$ and $e_A(\underline{(\cdot)}, \delta) = \{x \in A/\delta \subseteq \underline{(\cdot)}(x)\}$ respectively.

The set $DF_A(\bar{(\cdot)}, \underline{(\cdot)})_{(\gamma, \delta)} = \{x \in A/\gamma \subseteq \bar{(\cdot)}(x), \delta \subseteq \underline{(\cdot)}(x)\}$ is called a double framed including set of $\langle (\bar{(\cdot)}, \underline{(\cdot)}) : G \rangle$. It is clear that $DF_A(\bar{(\cdot)}, \underline{(\cdot)})_{(\gamma, \delta)} = i_A(\bar{(\cdot)}, \gamma) \cap e_A(\underline{(\cdot)}, \delta)$.

From now on, we will take G , as set of parameters, which is a group unless otherwise specified.

Note 1. Let $\lambda_S = (\bar{\alpha}_S, \bar{\beta}_S, E)$ be a double framed soft set over U . We will say that

$\lambda_S(e) = (\bar{\alpha}_S(e), \bar{\beta}_S(e))$ is image of parameter $e \in E$.

Definition 2.5. Let λ_A and $\lambda_B \in DFSE(U)$ then,

(i) If $\alpha_A(e) = \emptyset$ and $\beta_A(e) = U$ for all $e \in E$, α_A is said to be a null double-framed soft set, denoted by $\emptyset_b = (\emptyset, U, E)$.

(ii) If $\alpha_A(e) = U$ and $\beta_A(e) = \emptyset$ for all $e \in E$, α_A is said to be an absolute double-framed soft set, denoted by $\emptyset_b = (U, \emptyset, E)$.

(iii) λ_A is double-framed soft subset of λ_B , denoted by $\lambda_A \subseteq \lambda_B$, if $\alpha_A(e) \subseteq \alpha_B(e)$ and $\beta_A(e) \supseteq \beta_B(e)$ for all $e \in E$.

(iv) Double framed soft union and intersection of λ_A and λ_B , denoted by $(\alpha_A \cup \alpha_B) : A \cup B \rightarrow P(U)$ such that $(\alpha_A \cup \alpha_B)(e) = \alpha_A(e) \cup \alpha_B(e)$ and $(\beta_A \cap \beta_B)(e) = \beta_A(e) \cap \beta_B(e)$ for all $e \in E$.

Also $(\alpha_A \cap \alpha_B) : A \cap B \rightarrow P(U)$ such that $(\alpha_A \cap \alpha_B)(e) = \alpha_A(e) \cap \alpha_B(e)$ and $(\beta_A \cup \beta_B)(e) = \beta_A(e) \cup \beta_B(e)$ for all $e \in E$.

(v) Double framed soft complement of λ_A is denoted by λ^{c_A} and defined by $\lambda^{c_A} : E \rightarrow P(U) \times P(U)$ such that $\lambda^{c_A}(e) = \{(e, \alpha_A(e), \beta_A(e)) : e \in E\}$.

Definition 2.6 (20). A mapping $\mu : X \rightarrow [0,1]$, where X is an arbitrary non-empty set and is called fuzzy set in X .

Definition 2.7. Let $I : X \rightarrow L$ is called Double-framed fuzzy soft lattice over L if

(DFFSL1) $\alpha(x + y) \geq T\{\alpha(x), \alpha(y)\}$ (ii) $\alpha(-x) \geq \alpha(x)$

(DFFSL2) $\alpha(x \vee y) \geq T\{\alpha(x), \alpha(y)\}$ (iv) $\alpha(x \wedge y) \geq T\{\alpha(x), \alpha(y)\}$

(DFFSL3) $\beta(x + y) \leq S\{\beta(x), \beta(y)\}$ (vi) $\beta(-x) \leq \beta(x)$

(DFFSL4) $\beta(x \vee y) \leq S\{\beta(x), \beta(y)\}$ (iv) $\beta(x \wedge y) \leq S\{\beta(x), \beta(y)\}$ Where $(a, \beta) = (\mu, \gamma)$.

For all $x, y \in L$.

Definition 2.8. An Double-framed fuzzy set A in L is called Double-framed fuzzy quotient sublattice of L if, the following conditions are satisfied. (DFFQL1) $D_A(x + y) \geq$

$T\{D_A(x), D_A(y)\}$,

(DFFQL1) $D_A(-x) \geq D_A(x)$

(DFFQL1) $D_A(x * y) \geq \max\{\min\{D_A(x), D_A(y)\}\}$ for all $x, y \in L$.

Here $*$ represents the combination of meet and joint operations. Throughout this paper, the pair (α_A, β_A) is some times denoted as (μ_A, γ_A) for our convenient.

Definition 2.9. Let μ be a fuzzy subset of a set L and $t \in [0,1]$. Then the set $\mu_t = \{x \in L / \mu(x) \geq t\}$ is called level sub set of μ .

Definition 2.10. An Double-framed fuzzy soft quotient lattice A is said to be self-distributive double-framed fuzzy soft set in L and DA is fuzzy quotient lattice then $D_a^b(x) = a^b D(x)$, for any $a, b \in L$.

Definition 2.11. Let $f : L \rightarrow L^0$ be a lattice homomorphism. f is fuzzy lattice homomorphism if $f(x + y) = f(x) + f(y)$, for all $x, y \in L$.

Definition 2.12. A fuzzy set S dominates S^* if $S^* \supseteq S$. That is S^* dominates S .

Definition 2.13. Let μ and γ be two fuzzy quotient lattice. Then fuzzy quotient class is defined as $[\mu, \gamma]_{S^*}(x) = \max\{\mu_{S^*}(x), \gamma_{S^*}(x)\} = \{S^* \mu(x), S^* \gamma(x)\}$ for all $x, y \in L$.

3 PROPERTIES OF DOUBLE-FRAMED FUZZY

QUOTIENT SUBLATTICES

Proposition 3.1. If an double-framed fuzzy soft set A in L is a double-framed fuzzy soft Quotient sub lattice of L then so, is $A = \{\{x, \mu_A(x), 1 - \mu_A(x)\} / x \in L\}$.

Proof: Suppose A is an double-framed fuzzy soft quotient lattice of L . Then for any $x, y \in A$, $x + y \in A$ and $x * y \in A$.

$$\mu_A(x) = \mu_A(y) = 1 \text{ and } 1 - \gamma_A(x) = 1 - \gamma_A(y) = 0$$

Then

$$\mu_A(x + y) = T\{\mu_A(x), \mu_A(y)\} \mu_A(x * y) = \max\{\min\{\mu_A(x), \mu_A(y)\}\} \text{ and}$$

$$(1 - \gamma_A)(x + y) = S\{(1 - \gamma_A)(x), (1 - \gamma_A)(y)\}$$

$$(1 - \gamma_A)(x * y) = \min\{\max\{(1 - \gamma_A)(x), (1 - \gamma_A)(y)\}\}$$

Suppose $x, y \in L$ and atleast one of them say $y \notin A$, then

$$\mu_A(y) = 0, (1 - \gamma_A)(y) = 1, \gamma_A(x) \wedge \gamma_A(y) = 0$$

$$(1 - \gamma_A)(x) \vee (1 - \gamma_A)(y) = 1$$

$$\mu_A(x + y) \geq T\{\mu_A(x), \mu_A(y)\} \text{ and } \mu_A(x * y)_q = \max\{\min\{\mu_A(x, q), \mu_A(y, q)\}\}$$

$$(1 - \gamma_A)(x + y, q) \leq S\{(1 - \gamma_A)(x), (1 - \gamma_A)(y)\} \text{ and}$$

$$(1 - \gamma_A)(x * y) \leq \min\{\max\{(1 - \gamma_A)(x), (1 - \gamma_A)(y)\}\}.$$

Thus $(\mu_A, 1 - \gamma_A)$ satisfies the properties of double-framed fuzzy soft fuzzy quotient lattice. Conversely, Suppose $(\mu_A, 1 - \gamma_A)$ is fuzzy quotient lattice of L . Let $x, y \in A$. $\mu_A(x) = \mu_A(y) = 1, T\{\mu_A(x), \mu_A(y)\} = 1$. But, both $\mu_A(x + y)$ and $\mu_A(x * y) \geq T\{\mu_A(x), \mu_A(y)\}$. Thus, A is double-framed fuzzy soft quotient lattice of L .

Proposition 3.2. If an double-framed fuzzy soft set A in L is double-framed fuzzy quotient sub lattice of L if μ_A and γ_A^C are fuzzy lattice of L .

Proof: Let $I_A = (\mu_A, \gamma_A)$ be an double-framed fuzzy quotient lattice of L . Then obviously μ_A is fuzzy quotient lattice of L . Let $x, y \in L$. then,

$$(DFFQL1) \gamma_A^C(x + y) = 1 - \gamma_A(x + y) \geq 1 - \max\{\gamma_A(x), \gamma_A(y)\}$$

$$\geq T\{\gamma_A^C(x), \gamma_A^C(y)\}$$

$$(DFFQL2) \gamma_A^C(-x) = 1 - \gamma_A(-x) \geq 1 - \gamma_A(x) = \text{gamma}^C_A(x)$$

$$(DFFQL3) \gamma_A^C(x * y) = 1 - \gamma_A(x * y) \leq 1 - \min\{\max\{\gamma_A(x), \gamma_A(y)\}\}$$

$$= \max\{\min\{1 - \gamma_A(x), 1 - \gamma_A(y)\}\}$$

$$= \max\{\min\{\gamma_A^C(x), \gamma_A^C(y)\}\}$$

Conversely, suppose that μ_A and γ_A^C are double-framed fuzzy quotient lattice of L . Let $x, y \in L$. Then,

$$\begin{aligned}
 1 - \gamma_A(x + y) &= \gamma_A^C(x + y) \geq T\{\gamma_A^C(x), \gamma_A^C(y)\} \\
 &\geq T\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \geq 1 - \max\{\gamma_A(x), \gamma_A(y)\} \\
 1 - \gamma_A(-x) &= \gamma_A^C(-x) \geq \gamma_A(x) \text{ which imply } \gamma_A(x + y) \leq S\{\gamma_A(x), \gamma_A(y)\} \\
 \gamma_A(x) &\geq \gamma_A(y) \text{ Finally, for any } x, y \in L, \\
 1 - \gamma_A(x * y) &= \gamma_A^C(x * y) \geq \max\{\min\{\gamma_A^C(x), \gamma_A^C(y)\}\} \\
 &\geq \min\{\max\{1 - \gamma_A(x), 1 - \gamma_A(y)\}\}.
 \end{aligned}$$

Proposition 3.3. For any $t \in [0, 1]$, the maps U_t and V_t are surjective from $F(L)$ to $I(L) \cup [0, 1]$. Moreover, the quotient sets $F(L)/\mu$ and $F(L)/\gamma$ are equipotent to $I(L) \cup 0$.

Proof: Let $t \in [0, 1]$. Note that $0 = [0, 1]$ proof is in $F(L)$. Where 0 and 1 are fuzzy sets in L defined by $0(x) = 0$ and $1(x) = 1$ for all $x \in L$ obviously.

$$f_t(0) = U(0; t) = \phi = L(1; t) = g_t(0)$$

Let $J \notin \phi \in I(L)$. For $J = (X_j, \bar{X}_j) \in F(L)$, We have $f_t(J) = U(X_j; t) = J = L(\bar{X}_j; t) = g_t(J)$. Hence f_t and g_t are surjective.

Let f_t^* be a map from $F(L)/\mu$ to $I(L) \cup \{\phi\}$ defined by $f_t^*([I_A]_\mu) = f_t(I_A)$.

Assume that $U(\mu_A; t) = U(\mu_B; t)$ and $L(\gamma_A; t) = L(\gamma_B; t)$ for A, B is $F(L)$.

Then $A \mu B$ and $A \gamma B$, and hence $[I_A]_\mu = [I_B]_\mu$, $[I_A]_\gamma = [I_B]_\gamma$.

Therefore the maps f_t^* and g_t^* are injective.

Now let $J \in \phi \in I(L)$.

For $J = (X_j, \bar{X}_j) \in F(L)$, we have $f_t^*([J]_\mu) = f_t(J) = g_t([J]_\gamma)$. Finally, for $0 = [0, 1] \in F(L)$, we get $f_t^*([0]_\mu) = f_t(0) = U(0; t) = \phi = L(1; t) = g_t(0) = g_t([0]_\gamma)$. This shows that f_t^* and g_t^* are surjective.

Proposition 3.4. Let $\{M_t/t \in A \subseteq [0, 1]\}$ be a collection of quotient lattices of L such that

- (i) $J = \bigcup_{t \in A} M_t$
- (ii) For any $s, t \in A$, $s > t$ if and only if $M_s \subseteq M_t$.

Proof: Let $\{M_t/t \in A \subseteq [0, 1]\}$ be a collection of fuzzy quotient lattices of L . We consider the following two cases.

(i) $S = \sup\{t \in A / t < s\}$ and (ii) $S \notin \sup\{t \in A / t < s\}$

Case (i) implies that

$$x \in I_S \Leftrightarrow x \in M_t \text{ for all } t < S. \Leftrightarrow x \in \bigcap_{t < S} M_t \text{ when } I_S = \bigcap_{t < S} M_t \text{ which is a lattice of } L.$$

For the case (ii), there exists $\epsilon > 0$ such that $(S - \epsilon, S) \cap A = \emptyset$.

We claim that $I_S = \bigcup_{t \in A} M_t$, then $x \in M_t$ for some $t \geq S$.

It follows that $I_A(x) \geq t \geq S$ so that $x \in I_S$.

Conversely if $x \notin \bigcup_{t \in S} M_t$, then $x \notin M_t$ for some $t \geq S$,

Which implies that $x \notin M_t$ for all $t > S - \epsilon$, that is if $x \in M_t$ then $t \in S - \epsilon$. Thus

$I_A(x) \leq S - \epsilon$ and so $x \notin I_S$. Consequently, $I_S = \bigcup_{t \in S} M_t$ which is fuzzy quotient lattice

of L . This completes the proof.

Proposition 3.5. *Let A be an DFFS in L such that the non-empty upper and lower level sets $U(\mu_A; t)$ and $L(\gamma_A; t)$ of A are quotient lattices of L for every $t \in [0,1]$. Then A is a double-framed fuzzy quotient sub lattice of L .*

Proof: Let A be an DFFS in a lattice L . For any $x, y \in U(\mu_A; t)$. We have $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$ and for $x, y \in L(\gamma_A; t)$, we have $\gamma_A(x) \leq t$, $\gamma_A(y) \leq t$. Now

$$(DFFQL1) \mu_A(x+y) = T\{\mu_A(x), \mu_A(y)\} \geq T\{t, t\} \geq t, \quad \text{thus } x+y \in U(\mu_A; t)$$

$$\gamma_A(x+y) = S\{\gamma_A(x), \gamma_A(y)\} \leq S\{t, t\} \leq t, \text{ and thus } x+y \in L(\gamma_A; t)$$

$$(DFFQL2) \mu_A(-x) \geq \mu_A(x) \geq t, \quad \text{therefore } -x \in U(\mu_A; t) \text{ and}$$

$$\gamma_A(-x) \leq \gamma_A(x) \leq t, \quad \text{therefore } -x \in L(\gamma_A; t)$$

$$(DFFQL3) \mu_A(x * y) \text{ thus } x = \max\{\min\{I_A(x), I_A(y)\}\} \geq \max\{\min\{t, t\}\} \geq t,$$

$$* y \in U(\mu_A; t)$$

$$\gamma_A(x * y) = \min\{\max\{I_A(x), I_A(y)\}\} \leq \min\{\max\{t, t\}\} \leq t,$$

thus $x * y \in U(\mu_A; t)$

Hence A is a double-framed fuzzy quotient lattice in L .

Proposition 3.6. *If IFS A in L is an double-framed fuzzy quotient sub lattice then the non-empty upper and lower level sets $U(\mu_A; t)$ and $L(\mu_A; t)$ of A are lattices of L for every $t \in [0,1]$.*

Proof: Let μ be a Q -fuzzy quotient lattice of L and let $t \in [0,1]$. For any $x, y \in I_{At}$, we have $I_A(x+y) \geq T\{I_A(x), I_A(y)\} \geq t$ And so $x+y \in I_{At}$. Let $-x \in L$ and $x \in I_{At}$. Then $I_A(-x) \geq I_A(x) \geq t$. Let $x, y \in L$ we have

$I_A(x*y) \geq \max\{\min\{I_A(x), I_A(y)\}\} \geq \max\{\min\{t, t\}\} \geq t$, which shows that $x*y \in I_{At}$ Conversely, assume that I_{At} is a quotient lattice of L for every $t \in [0,1]$. If $I_A(x_0 + y_0) <$

$T\{I_A(x_0), I_A(y_0)\}$ for some $x_0, y_0 \in L$.

Then by taking

$$t_0 = \frac{1}{2} \{I_A(x_0 + y_0) + T\{I_A(x_0), I_A(y_0)\}\}$$

We have $I_A(x_0 + y_0) < t_0, I_A(x_0) > t_0$ and $I_A(y_0) > t_0$. Hence $x_0 + y_0 \notin I_{At_0}$, $x_0 \in I_{At_0}$ and $y_0 \in I_{At_0}$ This is a contradiction, and so $I_A(x+y) \geq T\{I_A(x), I_A(y)\}$, for all $x, y \in L$. Assume that $I_A(-x_0) < I_A(x_0)$ for some $x_0, y_0 \in L$.

Putting $M_0 = \frac{1}{2} \{I_A(-x_0), I_A(y_0)\}$, then $I_A(-x_0) < M_0 < I_A(x_0)$. It follows that $x_0 \in I_{Am_0}$ and $-x_0 \notin I_{Am_0}$ which is impossible. Hence $I_A(-x) \geq I_A(x)$ for all $x, y \in L$. If the condition (ii) of the definition (2.4) is not true, then for

$$\text{fixed } p_0 = \frac{1}{2} \{I_A(x * y) + I_A(x)\}$$

Then $I_A(x+y) \notin I_{p_0}$ and $x \in I_{p_0}$ and $y \in I_{p_0}$. This is a contradiction. Similarly, we can show lower level set are lattice in L for every $t \in [0,1]$.

Proposition 3.7. *If A is double-framed fuzzy quotient lattice of L , then the sets $L_{\mu_A} = \{x \in L / \mu_A(x) = \mu_A(0)\}$ and $Y_A = \{x \in L / \gamma_A(x) = \gamma_A(0)\}$ are lattices of L .*

Proof: A be an double-framed fuzzy quotient lattice and let $x, y \in L_{\mu_A}$. Then $I_A(x+y) \geq T\{I_A(x), I_A(y)\} = I_A(0)$ and so $I_A(x+y) = I_A(0)$ or $x+y \in L_{\mu_A}$. For every $-x \in L$ and $x \in L_{\mu_A}$, we have

$I_A(-x) \geq I_A(x) = I_A(0)$. Hence $-x \in L_{\mu_A}$, which shows that L_{μ_A} is a negative of L .

Let $x, y \in L_{\mu_A}$ and hence

$$I_A(x * y) \geq \max\{\min\{I_A(x), I_A(y)\}\} \geq \max\{\min\{I_A(0), I_A(0)\}\} = I_A(0).$$

Therefore, L_{μ_A} is a lattice of L . Similarly, we can show the complement of μ_A .

Proposition 3.8. *Let A be a self-distributive double-framed fuzzy sets in L . Then the IFS I_a^b in L is double-framed fuzzy quotient sub lattice of L for all $a, b \in L$.*

Proof: Since A be self-distributive double-framed fuzzy sets in L and I_A is fuzzy quotient lattice that is $I_a^b(x) = a^b I(x)$, for any $a, b \in L$.

$$(DFFQL1) \quad I_a^b(x + y) = a^b I(x + y) \geq T\{a^b I(x), a^b I(y)\} \geq T\{I_a^b(x), I_a^b(y)\}$$

$$(DFFQL1) \quad I_a^b(-x) = a^b I(-x) \geq a^b I(x) \geq I_a^b(x)$$

$$(DFFQL1) \quad I_a^b(x * y) = a^b I(x * y) \geq a^b \max\{\min\{I(x), I(y)\}\} \\ \geq \max\{\min\{a^b I(x), a^b I(y)\}\} \geq \max\{\min\{I_a^b(x), I_a^b(y)\}\}$$

Hence I_a^b is double-framed fuzzy quotient lattice in L .

Proposition 3.9. *Let L be a lattice and I be a sub set of L . If I is a double-framed quotient Q -fuzzy sub lattice of L , then the characteristic function ψ_I of I is a doubleframed fuzzy quotient sub lattices with respect to S .*

Proof: Since L be a lattice and $I \subset L$. The characteristic function of I is $\psi : L \rightarrow [0,1]$.

I is double-framed quotient sub lattice of L .

Claim: ψ_I is double-framed fuzzy quotient sublattice of L . for any $x, y \in L$.

$$(DFFQL1) \quad \psi_I(x + y) = \psi_I I(x + y) \geq \psi_I T\{I(x), I(y)\} \geq T\{\psi_I I(x), \psi_I I(y)\} \\ \geq T\{\psi_I(x), \psi_I(y)\}$$

$$(DFFQL1) \quad \psi_I(-x) = \psi_I I(-x) \geq \psi_I I(x) \geq \psi_I(x)$$

$$(DFFQL1) \quad \psi_I(x * y) = \psi_I I(x * y) \geq \psi_I \max\{\min\{I(x), I(y)\}\} \\ \geq \max\{\psi_I \{\min\{I(x), I(y)\}\}\} \geq \max\{\min\{\psi_I I(x), \psi_I I(y)\}\} \\ \geq \max\{\min\{\psi_I(x), \psi_I(y)\}\}$$

Therefore, ψ_I is fuzzy quotient sub lattice of L .

Proposition 3.10. *Let S be a S norm and μ, γ be a two double-framed quotient Q -Fuzzy sub lattice of L with respect to S . If S^* dominates S , then S^* -product, $[\mu, \gamma]_{S^*}$ of μ and γ is double-framed fuzzy quotient sub lattice of L .*

Proof: μ and γ be two double-framed fuzzy quotient sub lattice of L with respect to S -norms. Since S^* dominates the norm S .

Claim: S^* -product forms double-framed fuzzy quotient sub lattice of L .

$$\begin{aligned}
 (DFFQL1) \quad [\mu, \gamma]_{S^*}(x + y) &= \max\{\mu_{S^*}(x + y), \gamma_{S^*}(x + y)\} \\
 &\geq \max\{S^*\mu(x + y), S^*\gamma(x + y)\} \\
 &\geq \max\{S^*T\{\mu(x), \mu(y)\}, S^*T\{\gamma(x), \gamma(y)\}\} \\
 &\geq \max\{T\{S^*\{\mu(x), \gamma(x)\}, S^*\{\mu(y), \gamma(y)\}\} \\
 &\geq \max\{\{\mu_{S^*}(x), \gamma_{S^*}(x)\}, \{\mu_{S^*}(y), \gamma_{S^*}(y)\}\} \\
 &\geq T\{\max\{\mu_{S^*}(x), \gamma_{S^*}(x)\}, \max\{\mu_{S^*}(y), \gamma_{S^*}(y)\}\} \\
 &\geq T\{[\mu, \gamma]_{S^*}(x), [\mu, \gamma]_{S^*}(y)\}
 \end{aligned}$$

$$\begin{aligned}
 (DFFQL2) \quad [\mu, \gamma]_{S^*}(-x) &= \max\{\mu_{S^*}(-x), \gamma_{S^*}(-x)\} \\
 &\geq \max\{S^*\mu(-x), S^*\gamma(-x)\} \\
 &\geq \max\{S^*\mu(x), S^*\gamma(x)\} \\
 &\geq \{\mu_{S^*}(x), \gamma_{S^*}(x)\} \\
 &\geq [\mu, \gamma]_{S^*}(x)
 \end{aligned}$$

$$\begin{aligned}
 (DFFQL3) \quad [\mu, \gamma]_{S^*}(x * y) \\
 S^*\{\max\{\min\{\gamma(x), \gamma(y)\}\} \\
 &= \max\{\mu_{S^*}(x * y), \gamma_{S^*}(x * y)\} \\
 &\geq \max\{S^*\mu(x * y), S^*\gamma(x * y)\} \\
 &\geq \max\{S^*\{\max\{\min\{\mu(x), \mu(y)\}\}, \\
 &\geq \max\{\max\{S^*\{\min\{\mu(x), \mu(y)\}\}, \max
 \end{aligned}$$

$$\begin{aligned}
 &S^*\{\min\{\gamma(x), \gamma(y)\}\} \\
 &\geq \max\{\max\{\min\{S^*\mu(x), S^*\mu(y)\}, \\
 &\max\{\min\{S^*\gamma(x), S^*\gamma(y)\}\} \\
 &\geq \max\{\min\{\max\{S^*\mu(x), S^*\mu(y)\} \\
 &\max\{S^*\gamma(x), S^*\gamma(y)\}\} \\
 &\geq \max\{\min\{\max\{\mu_{S^*}(x), \gamma_{S^*}(x)\}, \\
 &\max\{\mu_{S^*}(y), \gamma_{S^*}(y)\}\} \\
 &\geq \max\{\min\{[\mu, \gamma]_{S^*}(x), [\mu, \gamma]_{S^*}(y)\}
 \end{aligned}$$

Therefore $[\mu, \gamma]_{S^*}(x * y) \geq \max\{\min\{[\mu, \gamma]_{S^*}(x), [\mu, \gamma]_{S^*}(y)\}$ if S^* dominates S .

Proposition 3.11. Let $f: R \rightarrow R^0$ be a homomorphism of R and R^0 . If μ and γ are two double-framed fuzzy quotient sub lattice of L^0 with respect to S then $f^1([\mu, \gamma]_{S^*})$ is double-framed fuzzy quotient sub lattice of L with respect to S .

Proof: A mapping $f: R \rightarrow R^0$ be a homomorphism μ and γ are double-framed fuzzy quotient sub lattice of L^0 with respect to S . Since S^* dominates S . We have $S^* \supseteq S$. **Claim:** $f^{-1}([\mu, \gamma]_{S^*})$ is double-framed fuzzy quotient sub lattice of L with respect to S .

$$\begin{aligned}
 (DFQQL1) \quad f^{-1}([\mu, \gamma]_{S^*})(x + y) &= [\mu, \gamma]_{S^*}f(x + y) \\
 &= [\mu, \gamma]_{S^*}\{fx + fy\} \\
 &= \max\{\mu_{S^*}(fx + fy), \gamma_{S^*}(fx + fy)\} \\
 &= \max\{S^*\mu(fx + fy), S^*\gamma(fx + fy)\} \\
 &\geq T\{\max\{S^*\{\mu(fx), \mu(fy)\}, S^*\{\gamma(fx), \gamma(fy)\}\} \\
 &\geq T\{\max(\mu_{S^*}fx, \mu_{S^*}fy),
 \end{aligned}$$

$$\begin{aligned}
 &\max(\gamma_{S^*}fx, \gamma_{S^*}fy)\} \\
 &\geq T\{\max(f^{-1}\mu_{S^*}(x), f^{-1}\gamma_{S^*}(x)),
 \end{aligned}$$

$$\begin{aligned}
 &\max(f^{-1}\mu_{S^*}(y), f^{-1}\gamma_{S^*}(y))\} \\
 &\geq T\{f^{-1}[\mu, \gamma]_{S^*}(x), f^{-1}[\mu, \gamma]_{S^*}(y)\}
 \end{aligned}$$

$$\begin{aligned}
 (DFQQL2) \quad f^{-1}[\mu, \gamma]_{S^*}(-x) &= [\mu, \gamma]_{S^*}f(-x) \\
 &= \max\{\mu_{S^*}f(-x), \gamma_{S^*}f(-x)\} \\
 &= \max\{f^{-1}\mu_{S^*}(x), f^{-1}\gamma_{S^*}(x)\}
 \end{aligned}$$

$$= \max\{f^{-1}S^*\mu(-x), f^{-1}S^*\gamma(-x)\}$$

$$\geq \max\{f^{-1}S^*\mu(x), f^{-1}S^*\gamma(x)\}$$

$$\geq \{f^{-1}\mu_{S^*}(x), f^{-1}\gamma_{S^*}(x)\} \geq f^{-1}[\mu, \gamma]_{S^*}(x)$$

$$\begin{aligned}
 (DFQQL3) \quad f^{-1}[\mu, \gamma]_{S^*}(x * y) &= [\mu, \gamma]_{S^*}f(x * y) = [\mu, \gamma]_{S^*}\{fx * fy\} \\
 &= \max\{\mu_{S^*}(fx * fy), \gamma_{S^*}(fx * fy)\} \\
 &= \max\{S^*\mu(fx * fy), S^*\gamma(fx * fy)\} \\
 &\geq \max\{\max\{\min\{S^*\mu(fx), S^*\mu(fy)\},
 \end{aligned}$$

$$\begin{aligned}
 &\max\{\min\{S^*\gamma(fx), S^*\gamma(fy)\}\} \\
 &\geq \max\{\min\{\max\{\mu_{S^*}fx, \mu_{S^*}fy\},
 \end{aligned}$$

$$\begin{aligned}
 &\max\{\gamma_{S^*}fx, \gamma_{S^*}fy\}\} \\
 &\geq \max\{\min\{\max\{\mu_{S^*}fx, \gamma_{S^*}fy\},
 \end{aligned}$$

$$\max\{\mu_{S^*}fx, \gamma_{S^*}fy\}\}$$

$$\begin{aligned} &\geq \max\{\min\{\max\{f^1\mu_{S^*}(x), f^1\nu_{S^*}(y)\}, \\ &\max\{f^1\mu_{S^*}(x), f^1\nu_{S^*}(y)\}\} \\ &\geq \max\{\min\{f^1[\mu.\nu]_{S^*}(x), f^1[\mu.\nu]_{S^*}(y)\}\} \end{aligned}$$

Therefore $f^1([\mu.\nu]_{S^*})$ is double-framed fuzzy quotient sub lattice of L under the domination of S^* .

Proposition 3.12. *A lattice homomorphic image of double-framed fuzzy quotient sub lattice of L with sup property is double-framed fuzzy quotient sub lattice.*

Proof: Let $f: L \rightarrow L^0$ be lattice homomorphism of L and let I_A be \mathcal{Q} -fuzzy quotient lattice of L with sup property. Given $x, y \in L$. We let $x_0 \in f^1(x^0)$ and $y_0 \in f^1(y^0)$ be

such that
$$I_A(x_0) = \sup_{h \in f^{-1}(x')} I_A(h), I_A(y_0) = \sup_{h \in f^{-1}(y')} I_A(h), \text{ respectively.}$$

Then we can deduce that

$$\begin{aligned} (DFQ1) \quad I_A^f(x^0 + y^0) &= \sup_{z \in f^{-1}(x^0 + y^0)} I_A(z) \geq T\{I_A(x_0), I_A(y_0)\} \\ &\geq T\{\sup_{h \in f^{-1}(x')} I_A(h), \sup_{h \in f^{-1}(y')} I_A(h)\} \end{aligned}$$

$$\geq T\{I_A^f(x'), I_A^f(y')\}$$

$$(DFQ1) \quad I_A^f(-x^0) = \sup_{z \in f^{-1}(-x^0)} I_A(z)$$

$$\geq I_A(x_0)$$

$$\geq \sup_{h \in f^{-1}(x')} I_A(h)$$

$$\geq I_A^f(x)$$

$$(DFQ1) \quad I_A^f(x' * y') = \sup_{z \in f^{-1}(x' * y')} I_A(z)$$

$$\geq \max\{\min\{I_A(x_0), I_A(y_0)\}\}$$

$$\geq \max\{\min\{\sup_{h \in f^{-1}(x')} I_A(h), \sup_{h \in f^{-1}(y')} I_A(h)\}\}$$

$$\geq \max\{\min\{I_A^f(x), I_A^f(y)\}\}, \quad \text{for all } x, y \in L.$$

Hence, lattice homomorphic image of fuzzy quotient lattice with sup-property forms double-framed fuzzy quotient sub lattice on L .

Applications:

Lattice structure has been found to be extremely important in the areas of quantum logic Ergodic theory, Reynolds operations, Soft Computing, Communication system, Information analysis system, artificial intelligences and physical science.

Conclusion:

(Saibaba, 2008) initiated the study of L -fuzzy lattice ordered groups and introduced the notion L fuzzy subgroups. (Goguen, 1967) replaced the valuation set $[0,1]$ by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L -fuzzy sets. K.V. Thomas and Latha. S. Nair studied Rough intuitionistic fuzzy sets in a lattice (Thomas et al, 2011). In this paper, we investigated the notion of double-framed fuzzy quotient lattice in a fuzzy lattice and then some basic properties are investigated.

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