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Peakon numerical solution of Cammasa-Holm non-linear equation

Amir Hossein Parand[1], Seyed Reza Mousavi[2], Samira Safajoo[3]
1- Corresponding author: Department of Civil Engineering, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran Email: a_h_parand@yahoo.com
2 - Department of Civil Engineering, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran Email: sr_mousavi_civil@yahoo.com
3 - Department of physics, science collage, Shiraz University, Shiraz, Iran

Abstract: In this paper peakon solitary waves solution of Cammasa-Holm non- linear equation obtained by numerical methods and peaked soliton solution are investigated by examining these waves interaction. The Cammassa- Holm is a non-linear third order equation because its peakon solutions has a discontinuity of derivative at its wave peak and cannot be solved using conventional numerical methods .Finite difference and Spectral methods are used to solve this equation .In finite difference method we used separated forms and upstream and downstream trick. In spectral Fourier method (by Fourier transformation), non-linear equation transform from physical space to Fourier space and in this new space becomes linear and can solve by one of common numerical methods. Eventually by inverse Fourier transformation, solutions transforms to physical space to observed. Soliton , peakon and anti peakon solutions of Cammasa- Holm equation can be illustrated as well by spectral method.
Keywords: numerical method, Cammasa-Holm equation, Finite difference, Spectral method, Soliton

Introduction

The discovery of water wave equation led to innovations in differential equations and theory of modeling and a lot of general information about theory of waves and solving differential equations generally obtained by water waves. What leads us to greater understanding of these waves was their direct use in mathematics and applied science and emanated from the fact that these waves and thus their interaction and characteristics easily is visible. Solitary waves are said to be solutions of nonlinear equations or systems that 1.represent a stable wave shape and 2.be localized; that were discovered by Russell [1].If the solitary waves after interacting with each other seprated without any changes they called soliton.

Only a few of nonlinear equations has a soliton solutions and due to their extensive use are considered. Peakon according to theory of integrable systems, is a soliton with the discontinuous first derivate (soliton with a sharp discontinuity in the wave peaks) which the wave shape is of the form $e^{-|x|}$[2]. An example of nonlinear equations which contains peakon solutions is

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx} \qquad (1)$$

Where b is constant.

This equation is integrable only for (b=2, Cammasa-Holm equation and b=3, Degasperis-Procesi equation) that are related to shallow waters and have peakon solution [4]. Cammasa - Holm equation can be written as a system of equations like

$$\begin{aligned} m_t &= -2mu_x - m_xu \\ m &= u - u_{xx} \end{aligned} \qquad (2)$$

Corresponding author: Mehdi Yadollahi, Email address: Yadollahi.Mehdi.PNU@gmail.com

Where m is momentum; or equivalent to a hyperbolic-elliptic system

$$
\begin{aligned}
& u_t + u u_x + p_x = 0 \\
& p - p_{xx} = u^2 + \frac{1}{2}(u_x)^2
\end{aligned} \tag{3}
$$

Where p is pressure or height of surface and u(x,t) represents height of free water surface above the flat surface.

Our goal is to create a numerical model to solve CH (Cammasa- Holm)equation to observe soliton stability and their interactions.

CH equation represent peakon solutions that are not continuous differentiable functions (derived peakon solution makes a delta function in the singularity) thus, inherently numerical modeling without losing a little accuracy is difficult. Also these equations includes both the second and third derivatives that makes numerical solution difficult. In the paper we use two numerical method to obtain peakon solitary waves solutions.

Finite Difference Method

CH equation of form (1) cannot directly be solved by the finite difference method that is due to third order derivatives and nonlinear terms specially $u_x u_x$ nonlinear term (by solving, this term caused an accumulation of errors and thus divergence of solution) and also is due to the special shape of peakon solution (finite difference method is based on numerical calculation that is according to befor and after point's derivative while peakons have discontinuous derivative at the peak) [6].

Thus we use seprated forms of equation. To solve, we write elliptical and hyperbolic separated forms of CH equation and for u>0, u<0 use upstream and downstream respectively.

Hence, any way wave's information is taken from where the wave is coming [7]. To write the equation as upstream or down stream form the following notation is used

$$
\begin{aligned}
& a \vee 0 = \max\{a, 0\} = \frac{a + |a|}{2} \\
& a \wedge 0 = \min\{a, 0\} = \frac{a - |a|}{2}
\end{aligned} \tag{4}
$$

Due to discontinuity of peakon solution, we use staggered lattice to write derivatives of equation in terms of differential formulas. Thus for the values of u, the information about midpoints of j+½ range is used, ultimately the equation is as follows

$$
\begin{aligned}
& \frac{d}{dt} u_{j+\frac{1}{2}} + (u_{j+\frac{1}{2}} \vee 0)\delta^- u_{j+\frac{1}{2}} + (u_{j+\frac{1}{2}} \wedge 0)\delta^+ u_{j+\frac{1}{2}} + \delta^+ p_j = 0 \\
& p_j - \delta^- \delta^+ p_j = (u_{j+\frac{1}{2}} \vee 0)^2 + (u_{j-\frac{1}{2}} \wedge 0)^2 + \frac{1}{2}(\delta^- u_{j+\frac{1}{2}})^2
\end{aligned} \tag{5}
$$

Where

$$
(\delta^+ u^n)_i = \frac{1}{h}(u_{i+1} - u_i) \qquad \delta = \frac{\delta^+ + \delta^-}{2} \tag{6}
$$

$$
(\delta^- u^n)_i = \frac{1}{h}(u_i - u_{i-1})
$$

Are forward, backward and central differential formulas. Numerical solution of the equation with below initial condition

$$u(x,0)=u_0(x)=e^{-|x|} \qquad (7)$$

Is a solitary peakon solution to the form of fig. 1.

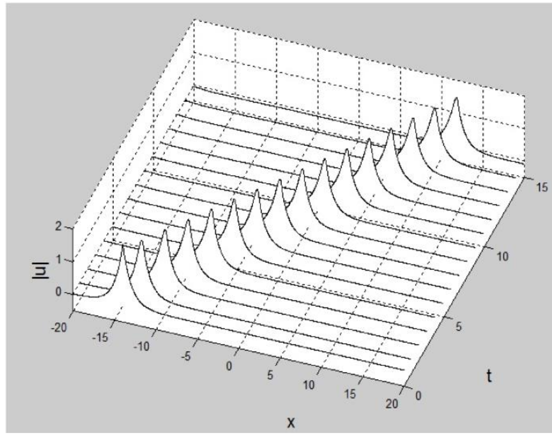

Fig1. Single peakon motion diagram

As you can see, peakon solitary waveform obtained from finite difference method does not change over time, but it's height changes, which seem to be due to numerical solution error. To investigate soliton we consider two peakon interactions.

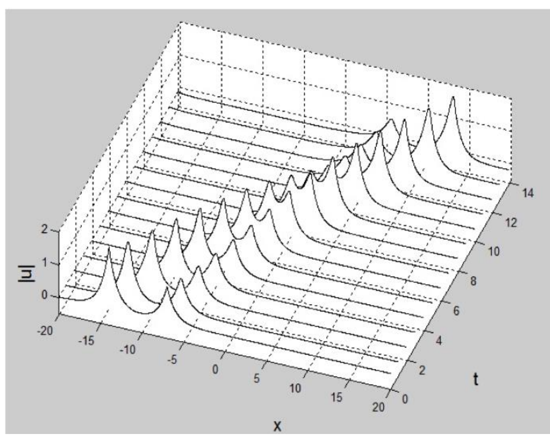

Fig2. Two peakon interaction

As it can be seen, peakons of CH equation are soliton. Looking carefully, we find that when two waves are crossing, some parasite gradually create which is caused by the accumulation of numerical errors and over time it begins to grow.

Finite difference method is not able to sustain the waveform in peakon and anti-peakon interaction. Hence we are looking for a method to have a less error in order to observe the peakon and anti-peakon interaction. For this purpose spectral method is used.

Spectral Method

Spectral method is one of the three major technology for numerical solution of partial differential equation and divided into two catagories of Fourier and Chebyshev spectral methods for alternative and non-alternative amplitudes respectively.

In spectral method by use of Fourier transformation the differential equation governing physical amplitude transforms to frequency amplitude or Fourier space [7]. By this transformation, the partial differential equation convert to an ordinary differential equation that becomes a much simple solution and can solve in frequency amplitude by existing numerical methods. Then by performing inverse Fourier transformation, transformed to physical amplitude which eventually leads to the discovery of problem' s solution [8]. Writing the Cammsa-Holm equation as

$$u_t + \frac{1}{2}(u^2)_x + \frac{\partial_x}{1-\partial_{xx}}\left(u^2 + \frac{1}{2}(u_x)^2\right) = 0 \qquad (8)$$

And then we apply Fourier transformation.Fourier transformation and it' s inverse operator acts as follows

$$\hat{u}_k = \mathscr{F}(u_j) = \sum_{j=0}^{N-1} u_j e^{-i\frac{2\pi k}{N}j} \qquad k = 0,1,2,\ldots,N-1 \quad (9)$$

$$u_j = \mathscr{F}^{-1}(\hat{u}_k) = \frac{1}{N}\sum_{k=1}^{N-1} \hat{u}_k e^{i\frac{2\pi k}{N}j} \qquad (10)$$

And for derivative's we have

$$\hat{u}'_j = \mathscr{F}\left(\frac{\partial^n u_j}{\partial x_j^n}\right) = (ik)^n \hat{u}_k \qquad (11)$$

By using above relations finally the equation becomes

$$\begin{aligned} &\frac{d}{dt}\hat{u}^N + \frac{ik}{2}\mathscr{F}\left(\left(\mathscr{F}^{-1}\hat{u}^N\right)^2\right) + \\ &\quad \frac{ik}{1+k^2}\mathscr{F}\left[\left(\mathscr{F}^{-1}\hat{u}^N\right)^2 + \frac{1}{2}\left(\mathscr{F}^{-1}\left(ik\hat{u}^N\right)\right)^2\right] = 0 \end{aligned} \qquad (12)$$

This equation in frequency space is an ordinary differential equation and we can solve it by forth order Runge-Kutta method with the initial condition (7).

The diagrams of peakon and also anti peakon solution are as follows; that are soliton solution.

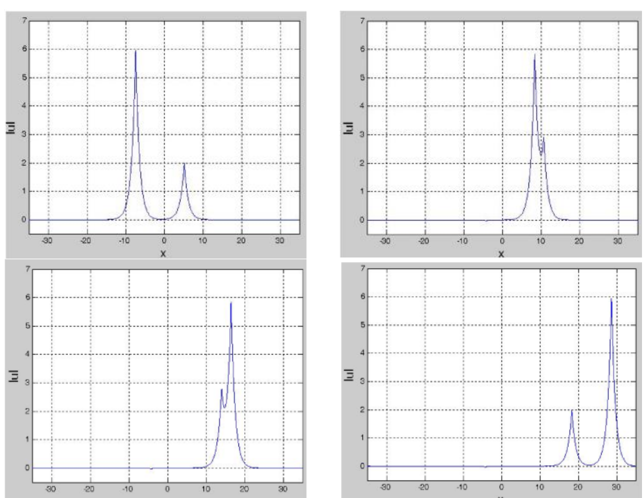

Fig3.interaction of two peakons by spectral method

By solving this method, the errors of finite difference method no longer exist and waves perfectly keep their forms and speed; in other words, they are solitons. The exact scrutiny of wave' s motion shows that first wave gives it' s energy to second wave when they reach to each other. An exchange of energy occurred between them.

Peakon and anti-peakon interaction that cannot be solved by the finite difference method; is solved by this method, and sixth order Runge-Kutta has been used to solve it.

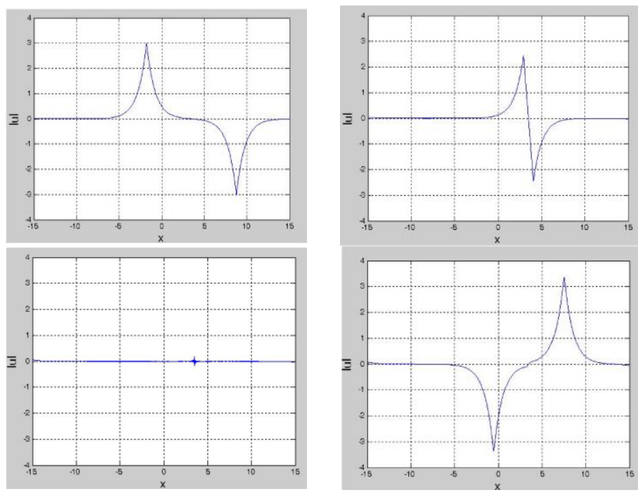

Fig 4.peakon and anti peakon interaction by spectral method

The ability of applying spectral method to solve this equation the solution after the solving of equation are clearly visible.

Conclusion

In this paper the peakon solution of travelling solitary wave for Cammasa-Holm equation was determined by both spectral and finite difference method and also we observed peaked soliton of equation through the interaction of two peakon solution. But due to special form of equation ,numerical error accumulation in finite difference method prevent from reaching to desired result about anti peaked soliton; that by applying Fourier spectral method this problem was resolved and all peaked and anti-peaked soliton solution of Cammasa-Holm equation can be seen with the minimum error.

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