Science Arena Publications
Specialty Journal of Engineering and Applied Science
ISSN: 2520-5943
Available online at www.sciarena.com
2019, Vol, 4 (2): 82-90

# Linear Preservers (strong) g- Tridiagonal and g- upper triangular majorization $\mathbb{R}^n$

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***Abstract:*** *In this article, we study the conditions that if there is a g- tridiagonaly doubly stochastic matrix for* $x = Ay. x. y \in \mathbb{R}^n$*, Then we call x as the g-tridiagonal majorized by y and represent all strong linear preservers of* $\prec$ *gut on* $\mathbb{R}^n$ *and in addition, we check that if* $X.Y \in M_{n.m}$ *are two elements of the set of* $n \times m$ *matrices, then there is the upper triangular g-row stochastic matrix R where it represents X = RY and some* $\prec$ *gut prerequisites on* $\mathbb{R}^n$ *and it reviews linear preservers (strong) of* $\prec$ *gut on* $\mathbb{R}^n$*.*

***Keywords:*** *g-tridiagonal majorized, Strong linear preservers, gut-majorization, g- row stochastic, g- doubly stochastic*

## INTRODUCTION

Majorization is a vital topic in mathematics and statistics that also plays a basic role in matrix theory and in many different fields; it has been a surprise over the past 25 years (Marshall, Olkin and Arnold, 2011). One of the primary topics of majorization is comparing equality and the difference in salary and income. The matrix A, g- row stochastic is a if the sum of each row A is 1. An $n \times n$ real matrix (not necessarily non-negative) A, g- doubly stochastic is, if the sum of each row and column is 1(This g- doubly stochastic matrix is the generalization of a doubly stochastic matrix).

For x, $x.y. \in \mathbb{R}^n$, it is said that x majorized by y (x $\prec$ y) if there a is exists doubly stochastic matrix D where x = Dy. For $x.y \in \mathbb{R}^n$, It is noted that x is g - *tridiagonal* majorized by y, (and written as $X \prec_{gt} Y$) if A there is a *tridiagonal* g- doubly stochastic $n \times n$ matrix, so that we show $x = Ay$. For $x.y \in \mathbb{R}^n$. x is gs- majorized by y when there is a g-doubly stochastic n×n matrix D, such that x = Dy, and with $X \prec_{gs} Y$ symbol. The X matrix is majorized by Y, if there is the *upper triangular g-row stochastic* matrix of R, such that $X = RY$ and we show it by with $X \prec gut\, Y$ symbol.

Suppose that n ($\mathbb{R}_n$) $\mathbb{R}^n$ is the vector space of all true vectors $(1 \times n)\, n \times 1$. If $T: \mathbb{R}^n \to \mathbb{R}^n$ and has a linear operator with relation ~ on $\mathbb{R}^n$), then it is said that T is the linear preserver when there is $x.y \in \mathbb{R}^n$

$$x \sim y \Longrightarrow T(x) \sim T(y)$$

And also T is a strong preserver ~, if,

$$x \sim y \Leftrightarrow T(x) \sim T(y)$$

The following symbols will be used throughoujt the article.
$\Omega_n$ is the set of all g-doubly stochastic matrices.
$\Omega_n^t$ is the set of all $n \times n$ tridiagonal g-doubly stochastic matrices.
J and e are matric and vector, respectively, with all of the elements equal to 1.
$e_i$ is the standard basic elements of $\mathbb{R}^n$ and

$$A_\mu = \begin{pmatrix} 1-\mu_1 & \mu_1 & & & 0 \\ \mu_1 & 1-\mu_1-\mu_2 & \mu_2 & \ddots & \mu_{n-1} \\ 0 & & & 1-\mu_1 & 1-\mu_1-1 \end{pmatrix}$$

Where $\mu = (\mu_1. \cdots . \mu_{n-1}) \in \mathbb{R}^{n-1}$. Since $\Omega_n^t$ is the set of all $n \times n$ tridiagonal g– doubly stochastic matrices and given that $A_\mu$ is tridiagonal, so it can be shown that

$$\Omega_n^t = \{A_\mu : \mu \in \mathbb{R}^{n-1}\}$$

$A^t$ is the transpose of the given A matrix.
For given vector $x \in \mathbb{R}^n. tr(x)$ is the sum of all components of x.
For a given linear operator, $T: \mathbb{R}^n \to \mathbb{R}^n$ represents the matrix T relative to the standard ordered basics $R^n$ by [T].
$\mathrm{R}_n^{gut}$ is the collection of all upper triangular g-row stochastic matrices.
e is a column of real vectors. With all of the entries equal to 1.

$N_k$ is the set $\{1. \cdots . k\} \subset N_k$

The card(S) is cardinality of the set S.
The Affine hull aff subset A of $\mathbb{R}^n$ is defined by

$$aff\ (A) := \left\{ \sum_{i=1}^{m} \lambda_i a_i \,|m \in \mathbb{N}. \sum_{i=1}^{m} \lambda_i = 1 . a_i \in A . \lambda_i \in \mathbb{R}. i \in \mathbb{N}_m \right\}$$

## 2. g- Tridiagonality majorization of $\mathbb{R}^n$

In this section, we describe the structure of strong linear preservers from $\prec gt$ on $\mathbb{R}^n$ and will be express in the context of the main theorem of this section (Armandnejad and Gashool, 2012; Hasani and Radjabalipour, 2006).

**Proposition 2.1** .Suppose $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear operator, in this case,T is the strong preserver of gs-majorization and only if for scalar non negative $\alpha \in \mathbb{R}$, $D \in \Omega_n$ be a invertible matrix, $T(x) = \alpha Dx$.

**Lemma 2.2**. If $T: M_{n.m} \to M_{n.m}$ is the linear operator, such that for $R.S \in M_m$ and invertible matrix $D \in \Omega_n$, $T(X) = DXR + JXS$ matrix Then T is a invertible If and only if $(R + nS)$ is invertible.
**Proof**. Without less of the general problem, assume $D = I$. If A is the representation of the matrix T with basic standards g-$M_{n.m}$, then we can easily see that A is the same as the following matrix,

$$\begin{pmatrix} R+ns & S & \cdots & S \\ 0 & R & \cdots & S \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{pmatrix}$$

In addition, T will be invertible if and only if R and (R + nS) are invertible.
We will explain the following result with the help of the above Lemma.

**Result 3.2**. Suppose $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is reversible, such that for $\alpha, \beta \in \mathbb{R}$ and the invertible matrix $T(x) = \alpha Dx + \beta Jx, D \in \Omega_n$. in this case,T is the invertible If and only if $\alpha(\alpha + n\beta) \neq 0$

**Lemma 4.2.** Suppose $x, y \in \mathbb{R}^n$, if both successive components of y are distinct, then $x \prec_{gt} y$ if and only if $tr(x) = tr(y)$.
**Proof.** If $x \prec_{gt} y$, so we put $y = (y_1, \cdots, y_j)^t$ and $x = (x_1, \cdots, x_j)^t$ for $(1 \leq j \leq n-1)$ and both consecutive components y are distinct, so $\sum_{i=1}^{j} x_j = \sum_{i=1}^{j} y_j$ that $tr(x) = tr(y)$. conversely, assume that both consecutive components of y are distinct. For each $(1 \leq j \leq n-1)$, we insert $\mu_j = \frac{\sum_{i=1}^{j}(x_i - y_i)}{y_{j+1} - y_j}$. Using a direct calculation and according to the preceding one, since Aμ. Tridiagonal g-doubly stochastic, so, we can see $x = A_\mu y$, where $\mu = (\mu_1, \ldots, \mu_{n-1})^t$ and therefore $x \prec_{gt} y$.
The following proposition gives an equivalent condition for $\prec_{gt}$ on $\mathbb{R}^n$.
**Theorem 2.5.**If x, y are two distinct vectors in $\mathbb{R}^n$and also suppose that $i_1 < i_2 < \ldots < i_k$ and

$$\{i_1, i_2, \ldots, i_k\} = \{j : 1 \leq j \leq n-1, y_j = y_{j+1}\}.$$

Then $x \prec_{gt} y$ if and only if for any $1 \leq l \leq k+1$,

$$\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$$

Where $i_0 = 0$ and $i_{k+1} = n$ .
**Proof.**. If $x \prec_{gt} y$, then $A_\mu \in \Omega_n^t$. So that $x = A_\mu y$, therefore for each $1 \leq j \leq n, x_j = \mu_{j-1}(y_{j-1} - y_j) + \mu_j(y_{j+1} - y_j) + y_j$
*where* $y_0 = \mu_0 = y_{n+1} = \mu_n = 0$. If for any $j \in \{i_1, i_2, \ldots, i_k\}$, $y_j = y_{j+1}$then we have for $1 \leq l \leq k+1$,

$$\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$$

Conversely, we put $y^1 = (y_1, y_2, \ldots, y_{i_1})^t$ and $x^1 = (x_1, x_2, \ldots, x_{i_1})^t$., in this case, both consecutive components of $y^1$ are distinct. Because $\sum_{i=1}^{j} x_j = \sum_{i=1}^{j} y_j$, so using lemma 4.2, $x^1 \prec_{gt} y^1$. Therefore $A_1 \in \Omega_{i_1}^t$, so that $x^1 = A_1 y^1$. Now we put $x^l = (x_{il-1} + 1, x_{il-1} + 2, \ldots, x_{il})^t$, $y^l = (y_{il-1} + 1, y_{il-1} + 2, \ldots, y_{il})^t$ for $2 \leq l \leq k+1$. Since that

$$\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$$

So using Lemma 4.2.$x^l \prec_{gt} y^l$, as a result, there is $A_l \in \Omega_{il-il-1}^t$ so that $x^l = A_l y^l$. Now we put $A := \oplus_{j=1}^{k} A_j$ This results in $A \in \Omega_n^t$ and $x = A_y$, and in this case $x \prec_{gt} y$.

**Lemma 6.2**: let $y \in \mathbb{R}^n$ also $i_1 < i_2 < \cdots < i_k$, $\{i_1, i_2, \ldots, i_k\} = \{j: 1 \leq j \leq n-1, y_j = y_{j+1}\}$, in this case $H_y := \{x \in \mathbb{R}^n: x \prec_{gt} y\}$ is an affine set with the dimension $n - (k+1)$.
**Proof.** Using Theorem 5.2 it can be concluded that for each $l$,

$$H_y = \left\{ x \in \mathbb{R}^n : \sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j \, , l \in \{1, \dots, k+1\} \right\}$$

Where $i_0 = 0$ and $i_{k+1} = n$. it is clear that $\lambda x + (1-\lambda)z \in H_y$ if $x, z \in H_y$. So, $H_y$ is an affine set $S \subseteq \mathbb{R}^n$ (called an affine set when for $x, y \in S$ and for every $\lambda \in \mathbb{R}$, $\lambda x + (1-\lambda)y \in S$), because every $x \in H_y$ must be match in the $k+1$ equation, then the Hy is an affine set with the dimension $dimH_y = n - (k+1)$.

**Result 2.7.** Suppose $y \in \mathbb{R}^n$, in this case $dimH_y = 0$ If and only if $y \in span\ (e)$.

**Proposition 8.2**. When $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the liner operator and T is a strong preserver of $\prec gt$, then the following statements are true:

1. T is invertible.
2. For each $i, j \in \{1, \dots, n\}$, we have $tr(Te_i) = tr(Te_j)$.
3. $Te \in span(e)$
4. [T] is a multiple of a g-doubly stochastic matrix.

**Proof**. (1). Suppose $T(x) = 0$. Since T is linear, therefore $T\,(0) = 0 = T\,(x)$. Hence it is obvious that $T(x) \prec_{gt} T\,(0)$. In this case, since T is a strong preserver for –gt majorization, so $x \prec_{gt} 0$. Consequently, there is a $R \in \Omega_n^t$ such that $x = R0$, and hence $x = 0$and because of this, T is invertible.

(2). Using the theorem 5.2, we have $e_j \prec_{gt} e_{j+1}$ for each$1 \leq j \leq n-1$. In this case,$Te_j \prec_{gt} Te_{j+1}$ for $1 \leq j \leq n-1$ and hence for $i, j \in \{1, \dots, n\}$,

$$tr(Te_i) = tr(Te_j)$$

(3). since T is invertible, so $a \in \mathbb{R}^n$ such that $Ta = e$. Using result 7.2, we have

$$dim(H_a) = dim(H_{Ta}) = 0$$

And hence $Te \in span\ (e)$.

(4). in (1) we proved that T and $T(x) = 0$ are invertible and in (2) the sum of all components $T(x)$ is expressed in relation to the standard basics $e_i$ and $e_j$. In this case [T] is a multiplicative of the g- doubly stochastic matrix.

Now we prove the main theorem of this section.

**Theorem 9.2.** Suppose $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear operator, then T is $a \prec_{gt}$ strong preserver if and only if $a, b \in \mathbb{R}$,So that $(a - b)\,(a + (n-1)\,b) \neq 0$ and [T] is one of the following matrices:

$$\begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{pmatrix}, \begin{pmatrix} b & b & \cdots & b & a \\ b & b & \cdots & a & b \\ \vdots & \vdots & & \vdots & \vdots \\ a & b & \cdots & b & b \end{pmatrix}$$

In other words, T is $a \prec_{gt}$strong preserver if and only if $\alpha, \beta \in \mathbb{R}$ such that $\alpha(\alpha + n\beta) \neq 0$ and $[T] = \alpha I + \beta J$or $[T] = \alpha P + \beta J$ where P is the backward identity matrix.

**Proof**. Let $A = [a_{ij}] = [T]$. If$n \leq 2$, then the notion $\prec_{gt}$ and $\prec_{gs}$ are the same on $\mathbb{R}^n$ and therefore, using Proposition 2.1 the proof is complete.

Let $n \geq 3$ without loss the generality. The fact that the conditions (1) and (2) of the preceding proposition are sufficient in order to T of a strong linear preserver of $\prec_{gt}$, can be proven according to the previous proposition. In that case, we must prove the necessity of the conditions.

Let T is a strong preserver of $\prec_{gt}$, then T is invertible using multicondition proposition 8.2. we put $\Phi := \{x \in \mathbb{R}^n : x \prec_{gt} e_1\}$. According to the theorem 5.2

$$\Phi = \{x \in \mathbb{R}^n : x_1 + x_2 = 1, x_3 = \ldots = x_n = 0\}$$

And $dim\Phi = 1$. Because T is a strong linear preserver, so

$$T(\Phi) = \{Tx \in \mathbb{R}^n : x \prec_{gt} e_1\} = \{T x \in \mathbb{R}^n : Tx \prec_{gt} T e_1\}$$

We have $dim\Phi = dimT(\Phi) = 1$ using the invertibility T. Since$Te_1 \in T(\Phi)$ and $Te_1, (n-1), dimT(\Phi) = 1$ has a similar consecutive component and hence for $b \in \mathbb{R}$, we have. $Te_1 = (b, \ldots, b, a_{n,1})^t$ or $Te_1 = (a_{1,1}, b, \ldots, b)^t$. We put $\Psi := \{x \in \mathbb{R}^n : x \prec_{gt} e_n\}$. From Theorem 5.2,

$$\Psi = \{x \in \mathbb{R}^n : x_{n-1} + x_n = 1, x_{n-2} = \ldots = x_1 = 0\}$$

And $dim\Psi = 1$. With the same argument we can prove for $c \in \mathbb{R}$ that $Te_n = (a_1, n, c \ldots c)^t$ or $Te_n = (c, \ldots, c, a_n, n)^t$. Now we consider all possible forms of $Te_1$ and $Te_n$.

Let $T(e_1) = (b, \ldots, b, a_n, 1)^t$. so,
$e_2 \prec_{gt} e_1 \Longrightarrow Te_2 \prec_{gt} Te_1$
$\Longrightarrow (a_{1,2}, a_{2,2}, \ldots, a_n, 2)^t \prec_{gt} (b, \ldots, b, a_{n,1})^t$
$\Longrightarrow a_{n-2,2} = \ldots = a_{2,2} = a_{1,2} = b$
$\Longrightarrow Te_2 = (b, \ldots, b, a_{n-1,2}, a_{n,2})^t$

For $e_{j+1} \prec_{gt} e_j$، $1 \le j \le n-1$. Using the same reasoning above, $Te_j = (b, \ldots, b, a_{n-j+1}, j, \ldots, a_{n,j})$. This result that

(1)

$$A = \begin{pmatrix} a_{1,1} & c & \cdots & c \\ & \ddots & & \vdots \\ * & & & c \\ & & & a_{n,n} \end{pmatrix}$$

Let T e1 ==T e١ = (a1,1, b, . . . , b)t can be shown in the same way:

(2)

$$A = \begin{pmatrix} b & & b & * \\ \vdots & \ddots & & \\ b & & & \\ b & \cdots & b & a_{n,n} \end{pmatrix}$$

Let $Te_n = (c, \ldots, c, a_{n,n})^t$ . in this case,
$e_{n-1} \prec_{gt} e_n \Rightarrow Te_{n-1} \prec_{gt} Te_n$
$\Rightarrow (a_{1,n-1}, a_{2,n-1}, \ldots, a_{n,n-1})^t \prec_{gt} (c, \ldots, c, a_{n,n})^t$
$\Rightarrow a_{1,n-1} = a_{2,n-1} = \ldots = a_{n-2,n-1} = c$
$\Rightarrow Te_{n-1} = (c, \ldots, c, a_{n-1,n-1}, a_{n,n-1})$
For $2 \le i \le n-3$, $e_{n-i} \prec_{gt} e_{n-i+1}$. In this case we have a similar argument above $Te_i = (c, \ldots, c, a_{i,i}, \ldots, a_{n,i})^t$. this results that

(3)

$$A = \begin{pmatrix} a_{1,1} & c & \cdots & c \\ & \ddots & & \vdots \\ * & & & c \\ & & & a_{n,n} \end{pmatrix}$$

Now let's assume $Te_n = (a_{1,n,c}, \ldots, c)^t$. It can be shown in the same way

(4)

$$A = \begin{pmatrix} * & & & a_{1,n} \\ & & & c \\ & & & \vdots \\ a_{n,1} & c & \cdots & c \end{pmatrix}$$

Since $n \geq 3$ and T are invertible, only possible modes are (4), (1) and (3), (2). According to Theorem 5.2, A is a multiple of g-doubly stochastic matrix. In this case A has one of the following forms:

$$\begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{pmatrix}, \begin{pmatrix} b & b & \cdots & b & a \\ b & b & \cdots & a & b \\ \vdots & \vdots & & \vdots & \vdots \\ a & b & \cdots & b & b \end{pmatrix}$$

In both cases, we use result 3.2 to obtain $(a - b)(a + (n - 1)b) \neq 0$. Theorem will prove in this case.

**3. Gut – majorization on $\mathbb{R}^n$**

In this section, we express equivalent condition for the gut-majorization $\mathbb{R}^n$ and some prerequisites $\prec_{gut}$on $\mathbb{R}^n$. In addition, we specify all the linear strong preservers of $\prec_{gut}$ on $\mathbb{R}^n$ (Armandnejad and Ilkhanizadeh Manesh, 2012).

The following point proves a criterion for gut-majorization on $\mathbb{R}^n$.

**Tip 3.1**. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $l = min\,\{i \mid y_i = y_{i+1} = \ldots = y_n\}$

In this case, $x \prec_{gut} y$ if only and if $x_l = x_{l+1} = \ldots = x_n = y_n$. In other words $x \prec_{gut} y$ if only and if for $i \in \mathbb{N}_n$.

**Lemma 3.2**. Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is linear preserver of $\prec_{gut}$. If $S: \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ is linear function, so that $[S] = [T]\,[2, 3, \ldots, n]$then S preserver $\prec_{gut}$ is on $\mathbb{R}^{n-1}$.

**Proof.** Let $x' = (x_2, \ldots, x_n)^t, y' = (y_2, \ldots, y_n)^t \in \mathbb{R}^{n-1}$ if only if $x' \prec_{gut} y'$, in this case using point 3.1

$$x := (0, x_2, \ldots, x_n)^t \prec_{gut} y := (0, y_2, \ldots, y_n)^t$$

Hence $T_x \prec_{gut} T_y$ Which requires $Sx' \prec_{gut} Sy'$ then S is $\prec_{gut}$preserver on $\mathbb{R}^{n-1}$.

**Lemma 3.3**. If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear preserver $\prec_{gut}$, then [T] is an upper triangular matrix.

**Proof**. We prove by induction on n. Let$[T] = [a_{ij}]$. Consider x, y as $x = e_1$ and $y = 2e_1$ because $x \prec_{gut} y$, so $e_1 \prec_{gut} 2e_1$ and $Tx \prec_{gut} Ty$, that $a_{21} = 2a_{21}$ and so $a_{21} = 0$. We assume for $n > 2$ the matrix representation of each linear preserver of $\prec_{gut}$on $\mathbb{R}^{n-1}$ is the upper triangular matrix. If $S: \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ is a linear function where $[S] = [T]\,[2, 3, \ldots, n]$ using Lemma 3.2 is the linear function S of preserver $\prec_{gut}$ on $\mathbb{R}^{n-1}$. We deduced from the induction hypothesis that $[S]$ on $(n-1) \times (n-1)$ is an upper triangular matric. Now, we need to show that $a_{21} = a_{31} = \cdots = a_{n1} = 0$ . We select $x = e_1$and $y = e_2$. In this case, $x \prec_{gut} y$ is, matrix thus

$$\mathrm{Tx} = (\mathrm{a}_{11}, \mathrm{a}_{21}, \ldots, \mathrm{a}_{n-11}, \mathrm{a}_{n1})^{\mathrm{t}} \prec_{gut} (\mathrm{a}_{12}, \mathrm{a}_{22}, 0, \ldots, 0)^{\mathrm{t}} = \mathrm{Ty}$$

Using point 3.1, we have $a_{31} = a_{41} = \cdots = a_{n1} = 0$. Now we have to prove $a_{21} = 0$. We prove it to the Proof by contradiction. If $a_{21} \neq 0$ is possible, for $x = e_1$ and $y = \left(\frac{-a_{22}}{a_{21}}, 1, \ldots, 0\right)^t$ that $x \prec_{gut}\, y and\ so\ Tx \prec_{gut} Ty.$ It is concluded that $a_{21} \in aff\{0\}$, where $a_{21} = 0$is a contradiction. In this case, $a_{21} = 0$. Therefore, the induction hypothesis is complete. So $[T]$ is an upper triangular matrix.

**Lemma4.3.** Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the linear function for each $a_{kt} \neq 0, k, t \in \mathbb{N}_n$ and $[T] = [a_{ij}]$ where $a_{k+1t} = \cdots = a_{nt} = 0$. Therefore, there is j where $(1 + t \leq j \leq n - 1)$ such that $a_{k+1j} = \cdots = a_{nj} = 0$, then isn't $\prec_{gut}$ preserver.

**Proof**. T will be $\prec_{gut}$ preserver if and only if for any $\alpha \in \mathbb{R} - \{0\}$,$\alpha T$ is $\prec_{gut}$ preserver. Without loss total problem, assume that $a_{kt} = 1$. Considering $x = e_t$ and $y = -a_{kj}\, e_t + e_j$, we see that $x \prec_{gut} y$, but $Tx \nprec_{gut} Ty$. In this case, $\prec_{gut}$ does not T preserver.

The following theorem below is a structure of all the linear functions that defines the $\prec_{gut}$ preserver.

**Theorem 5.3.**If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear function, then T is the $\prec_{gut}$ preserver if and only if one of the following choices is true:

(1)$Te_1 = \ldots = Te_{n-1} = 0$. In other words

$$(T) = \begin{pmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & 0 & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

(2) There is T where $1 \leq i_1 < \ldots < i_m \leq n - 1$ and $1 \leq t \leq n - 1$ so that $a_{i_1 t}, a_{i_1 t+1}, \ldots, a_{i_m n-1} \neq 0$

$$[T] = \begin{pmatrix} 0 & * & & & & \\ & a_{i_1 t} & & & * & \\ & & \cdots & a_{i_2 t+1} & \cdots & \\ & 0 & & & a_{i_m n-1} & \\ & & & & 0 & * \end{pmatrix}$$

And $r_{i_k} \in aff\,\{r_{i_{k+1}}, \ldots, r_n\}$ for each $k \in \mathbb{N}_m$ where $r_i$ is the sum of all the arrays on the i-th row of [T].

**Proof**. To begin the proof, we first prove the conditions. If (1) is established then it is clear that T is $\prec_{gut}$ preserver. Let (2) is established that prove by induction on n. If $n = 2$, the proof is clear. Let $n \geq 3$. The result is for all linear functions on $\mathbb{R}^{n-1}$ that are proven by conditions in hypothesis. Let $y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)^t$ so that $x \prec_{gut} y$, suppose that $Tx \prec_{gut} Ty$. For this propose, linear function $S: \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ where take $[S] = [T][2, 3, \ldots, n]$ and put. In $x' \prec_{gut} y'$, we apply the inductive hypothesis for S and. $Sx' \prec_{gut} Sy'$.

$$((Tx)_2, \ldots, (Tx)_n)^t \prec_{gut} ((Ty)_2, \ldots, (Ty)_n)^t$$

In this case, it suffices to show that

$$(Tx_1) \in aff\,\{(Ty)_1, \ldots, (Ty)_n\}$$

We put

$$I := \{i \in \mathbb{N}_n : a_{ii} = \ldots = a_{in-1} = 0\}$$

We consider the proof with the following two conditions:

Condition a) $rd\,\{a_{in} | i \in I\} \geq 2$ .

If $y_n \neq 0$, so

$(\mathrm{Tx})_1 \in \mathrm{aff}\,\{(\mathrm{Ty})_1, \ldots, (\mathrm{Ty})_n\} = \mathbb{R}$

If $y_n = 0$, then the statement $card\{(Ty)_1,\ldots,(Ty)_n\} \geq 2$ is established. In other words, $y_n = 0$ and $(Ty)_1\ ,\ldots,(Ty)_n\ = 0$. Since $a_{i_m n-1} \neq 0$, $(Ty)_{im} = 0$, It follows that $y_{n-1} = 0$. So we see that $y_{n-2} = \ldots = y_t = 0$. Hence Using the point 1.3. $x_t = \ldots = x_n = 0$, in this case, $(Tx)_1 = (Ty)_1 = 0$ and the proof of condition (a) is complete.

Condition b) $card\ \{a_{in} | i \in I\}\ <\ 2$.

If $card\ \left\{\left(T_y\right)_1,\ldots,(T_n)_n\right\} \geq 2$, then poof is complete. Otherwise $(Ty)_{im} = (Ty)_n$, because

$$r_{i_m} \in aff\ \left\{r_{i_m+1},\ldots,r_n\right\}$$

And $y_{n-1} = y_n$ and $a_{i_m n-1}$ and We can conclude in the same way that $y_t = \cdots = y_{n-2}\ = y_n$, so using point 1.3, $x_t = \cdots = x_n$, then $(Tx)_1 = (Ty)_1$ that is a desirable result. Now we have to prove that T is preserver $\prec_{gut}$and (1) isn't established. We have to show that (2) is established and use induction for n. For $n = 2$, it is easy to prove. In this case, we assume that $n \geq 3$ and the linear preserver $\prec_{gut}$ is on 1-Rn. If $S\colon \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ is a linear function, then $[S] = [T]\ [2, 3\ldots, n]$.

Using lemma 3.2, S is the preserver $\prec_{gut}$ is on $\mathbb{R}^{n-1}$. We now apply the inductive hypothesis for S, and we divide the proof into two stages.

Condition c) S is in (1). Using lemma 4.3, the first non-zero column of T should be the $n-1$ column. In this case, we must show that

$$r_1 \in aff\{r_2,\ldots,r_n\}$$

Where we have this in proof of lemma 4.3. We can assume that $a_{1n-1} = 1$,

$$y\ = (a_{nn} - a_{1n})\, e_{n-1} + e_n, x = e_{n-1} + e_n$$

It is obvious that $x \prec_{gut} y$ and therefore $Tx \prec_{gut} Ty$ yields that

$$r_1 \in aff\{r_2,\ldots,r_n\}$$

Condition d) S is true in (2). If column, $1,\ldots,t_1$ of T is zero, then we have nothing to prove, Otherwise, using Lemma 4.3, first non-zero column of T should be the $n-1$ column. It is enough we show that

$$r_1 \in aff\{r_2,\ldots,r_n\}$$

Similarly, from condition (i) we can assume that $a_{1t-1} = 1$ and we give

$$y = \left(-\sum_{i=t}^{n} a_{1i} + a_{nn}\right) e_{t-1} + \sum_{i=t}^{n} e_i, x = \sum_{i=t=1}^{n} e_i$$

And it is concluded that

$$r_1 \in aff\{r_2,\ldots,r_n\}$$

**Lemma 6.3.** If $T\colon M_{n,m} \longrightarrow M_{n,m}$ is the linear function and T is the strong preserver $\prec_{gut}$, then T is invertible.

**Proof.** if $X \in M_{n,m}$ and $TX = 0$, since $TX = T0$ and T is the strong preserver $\prec_{gut}$, then $X = 0$,so T is invertible.

The following theorem specifies all of the linear functions $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, which is a gut-majorization strong preserver.

**Theorem 7.3**. The linear function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the strong preserver $\prec_{gut}$ if and only if there is a invertible matrix $A \in \mathbb{R}^n$ such that $[T] = \alpha A$ that $\alpha \in \mathbb{R} - \{0\}$.

**Proof**. Let T is a strong preserver $\prec_{gut}$. Then, according to Lemma 6.3 T is invertible and respect to lemma 3.3, $a_{11} \neq 0$, and apply theorem 5.3 and get the desired result.

Now we can get the following result from this theorem.

**Result 8.3**. If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the preserver $\prec_{gut}$, then T is the strong preserver of $\prec_{gut}$ if and only if T is invertible.

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