

The deformation retract of time like helix in Minkowski 3-space

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ABSTRACT: Our aim in the present article is to introduce and study new types of retraction of time like helix in Minkowski 3-space. Types of the deformation retracts of the time like helix in Minkowski 3-space are presented. The relations between the retraction and the deformation retract of time like helix are deduced. Types of minimal retraction of time like helix in Minkowski 3-space are also presented . New types of homotopy maps are described.

Keywords: Minkowski 3-space, time like helix, retractions, deformation retracts.

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Introduction

As is well known, the theory of deformation retracts is always one of interesting topics in Euclidian and Non-Euclidian space and it has been investigated from the various viewpoints by many branches of topology and differential geometry [1-21].

Helix is one of the most fascinating curves in Science and Nature .From the view of differential geometry ,a helix is a geometry curve with non –vanishing constant curvature (or first curvature of the curve and denoted by k_1) and non-vanishing constant torsion (or second curvature of the curve and denoted by k_2). Indeed a helix is a special case of the general helix .A curve of constant slope or general helix in Euclidean 3-space E^3 ,is defined by the property that the tangent makes a constant angle with a fixed straight line [14-17]. It is well known that to each unit speed curve $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ in the Euclidean space \mathbb{R}^3 whose successive derivatives $\alpha'(s)$, $\alpha''(s)$ and $\alpha'''(s)$ are linearly independent vectors ,one can associate three mutually orthogonal unit vector field T,N and B called respectively the tangent ,the principle normal and the binormal vector field [22] .At each point $\alpha(s)$ of curve α ,the planes spanned by $\{T,N\}$, $\{T,B\}$ and $\{N,B\}$ are known respectively as the osculating plane ,the rectifying plane and the normal plane [1].The curves $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ for which the position vector α always lie in their rectifying plane ,are simplicity called rectifying curves .Similarly ,the curves for which the position vector α always lie in their osculating plane , are for simplicity called osculating curves. Moreover , the curves for which the position vector α always lie in their normal plane , are for simplicity called normal curves [23]. The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by $g = -dx_1^2 + dx_2^2 + dx_3^2$, Where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3 .Since g is an indefinite metric ,recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters ,it can be space like if $g(v,v) > 0$ or $v = 0$, Time like if $g(v,v) < 0$ and light like if $g(v,v) = 0$ and $v \neq 0$.Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be space like ,time like or light like ,if all of its velocity vectors $\alpha'(s)$ are respectively , space like ,time like or light like respectively [1, 24]. Minkowski space is originally from the relativity in physics .In fact ,a time like curve corresponds to the path of an observer moving at less than the speed of light , a light like curves correspond to moving at the speed of light and a space like curves moving faster than light [25]. Consider the moving Frenet frame $\{T,N,B\}$ along the curve $\alpha(s)$ in E_1^3 . For an arbitrary curve $\alpha(s)$ in the space E_1^3 ,the following Frenet formula

are given in [26]. A curve in Lorentzian space L^n is a smooth map $\alpha : I \rightarrow L^n$ where I is the open interval in the real line \mathbb{R} . The interval I has a coordinate system consisting of the identity map u of I . The velocity of α at $t \in I$ is $\alpha' = \frac{d\alpha(u)}{du} \Big|_t$. A curve α is said to be regular if $\alpha'(t)$ does not vanish for all t in I . $\alpha \in L^n$ is space like if its velocity vectors α' are space like for all $t \in I$, similarly for time like and null. If α is a null curve, we can reparametrize it such that $\langle \alpha'(t), \alpha'(t) \rangle = 0$ and $\alpha'(t) \neq 0$ [27]. If α is time like curve, then the Frenet formula read

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & K_1 & 0 \\ K_1 & 0 & K_2 \\ 0 & -K_2 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Where $g(T, T) = -1, g(N, N) = 1, g(B, B) = -1, g(T, N) = 0, g(T, B) = 0$ and $g(N, B) = 0$.

Definitions

We adapted these definitions and concepts to fit coherently into the framework of this article.

i- A subset A of a topological space X is called a retract of X if there exists a continuous map

$r : X \rightarrow A$ called a retraction such that $r(a) = a$ for any $a \in A$ [2,4 5,6].

ii- A subset A of a topological space X is a deformation retracts of X if there exists a retraction

$r : X \rightarrow A$ and a homotopy $\varphi : X \times I \rightarrow X$ such that:

$$\left. \begin{aligned} \varphi(x, 0) &= x \\ \varphi(x, 1) &= r(x) \end{aligned} \right\} x \in X$$

$\varphi(a, t) = a, a \in A, t \in [0, 1]$ [3,5].

Main Results

Let $\alpha(s)$ be a time like helix. Then we can write its position vector as follows:

$$\alpha(s) = \lambda(s)T(s) + \mu(s)N(s) + \nu(s)B(s) \tag{1}$$

Differentiating equation (1) with respect to s and using Frenet formula we have

$$\lambda'(s) + \mu(s)K_1 = 1, \lambda(s)k_1 + \mu'(s) - \nu(s)K_2 = 0, \mu(s)K_2 + \nu'(s) = 0 \tag{2}$$

From equation (2) we have

$$\mu'' - \mu(s)(K_1^2 - K_2^2) + K_1 = 0 \tag{3}$$

Now we are going to discuss the following case

If $K_1^2 < K_2^2$. The solution of equation (3) is

$$\mu(s) = C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \tag{4}$$

Where $C_1, C_2 \in \mathbb{R}$. From equation (2) we have $\lambda'(s) = 1 - \mu(s)K_1$. By using (4) we find the solution of this equation as follows:

$$\lambda(s) = S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \tag{5}$$

Also from (2) we have $\nu'(s) = -\mu(s)K_2$. By using (4) we get

$$\nu(s) = -\frac{C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \tag{6}$$

Hence, the position vector $\alpha_2(s)$ is

$$\alpha_2(s) = \left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) +$$

$$\left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2}\right) B(s).$$

Now we are going to discuss the retraction of the position vector $\alpha_2(s)$ as follows

Let $r_i : \{\alpha_2(s) - \beta\} \rightarrow \{\alpha_2(s) - \beta\}^*$, where $\{\alpha_2(s) - \beta\}$ be the open helix and $\{\alpha_2(s) - \beta\}^*$ be the retraction of the position vector $\alpha_2(s)$.

Now we discuss the following cases:

If $C_1=0$, we have the following retraction defined as:

$$r_1(\alpha_2(s)) = \left(S + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2}\right) T(s) + \left(C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}\right) N(s) + \left(\frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2}\right) B(s).$$

Also, if $C_2=0$ we obtain the following retraction defined by:

$$r_2(\alpha_2(s)) = \left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2}\right) T(s) + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}\right) N(s) + \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2}\right) B(s).$$

If $K_1 = 0$, we present the following retraction given by $r_3(\alpha_2(s)) = (S) T(s) + (C_1 \cos(S K_2) + C_2 \sin(S K_2)) N(s) + (-C_1 \sin(S K_2) + C_2 \cos(S K_2)) B(s)$.

If $K_2 = 0$ we have the following retraction defined as:

$$r_4(\alpha_2(s)) = \left(S - \frac{C_1 K_1}{\sqrt{-K_1^2}} \sin(S\sqrt{-K_1^2}) + \frac{C_2 K_1}{\sqrt{-K_1^2}} \cos(S\sqrt{-K_1^2}) - S\right) T(s) + \left(C_1 \cos(S\sqrt{-K_1^2}) + C_2 \sin(S\sqrt{-K_1^2}) + \frac{1}{K_1}\right) N(s).$$

If $T(s) = 0$, we present the following retraction given by:

$$r_5(\alpha_2(s)) = \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}\right) N(s) + \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2}\right) B(s).$$

Also, if $N(s) = 0$ we obtain the following retraction defined as:

$$r_6(\alpha_2(s)) = \left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2}\right) T(s) + \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2}\right) B(s).$$

Now, if $B(s) = 0$ we present the following retraction given by:

$$r_7(\alpha_2(s)) = \left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2}\right) T(s) + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}\right) N(s).$$

Also, if $T(s) = N(s) = 0$, we have the following retraction defined as:

$$r_8(\alpha_2(s)) = \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2}\right) B(s).$$

Now, if $T(s) = B(s) = 0$ we present the following retraction given by:

$$r_9(\alpha_2(s)) = \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}\right) N(s).$$

Also, if $N(s) = B(s) = 0$, we have the following retraction defined as:

$$r_{10}(\alpha_2(s)) = \left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2}\right) T(s).$$

Moreover, if $T(s) = N(s) = B(s) = 0$, we have the following minimal retraction defined as:

$$r_{11}(\alpha_2(s)) = \{0, 0, 0\}.$$

In this position ,we present some cases of deformation retract of time like helix in Minkowski 3-space .The deformation retract of time like helix is

$$\varphi: \{\alpha_2(s) - \beta\} \times I \rightarrow \{\alpha_2(s) - \beta\},$$

where $\{\alpha_2(s) - \beta\}$ be the open helix of the position vector $\alpha_2(s)$ and I is the closed interval $[0,1]$,be present as

$$\begin{aligned} \varphi(x, h): & \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} N(s) \right) \\ & \left. + \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) - \beta \times I \rightarrow \\ & \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} N(s) \right) \\ & \left. + \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) - \beta). \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_1(\alpha_2(s))$ is

$$\begin{aligned} \varphi(x, h) = & (1 - h) \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} N(s) \right) \\ & \left. + \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) - \beta \\ & + h \left(\left(S + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} N(s) + \left(\frac{-C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) \end{aligned}$$

Where

$$\begin{aligned} \varphi(x, 0) = & \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} N(s) \right) \\ & \left. + \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) - \beta, \end{aligned}$$

and

$$\begin{aligned} \varphi(x, 1) = & \left(\left(S + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} N(s) + \left(\frac{-C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_2(\alpha_2(s))$ is defined as

$$\begin{aligned} \varphi(x, h) = & \cos \frac{\pi h}{2} \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} N(s) \right) \\ & \left. + \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) - \beta \end{aligned}$$

$$+ \sin \frac{\pi h}{2} \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) + \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right).$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_3(\alpha_2(s))$ is defined by

$$\begin{aligned} \varphi(x, h) = & \frac{1-h}{1+h} \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) \cdot \beta \quad \left. \right) + \frac{2h}{1+h} \quad ((S) T(s) \\ & + (C_1 \cos(S K_2) + C_2 \sin(S K_2)) N(s) + \\ & (-C_1 \sin(S K_2) + C_2 \cos(S K_2)) B(s)). \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_4(\alpha_2(s))$ is given by

$$\begin{aligned} \varphi(x, h) = & \cos \frac{\pi h}{2} \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) - \beta + \sin \frac{\pi h}{2} \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \right. \right. \\ & \left. \left. \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) \right) \cdot S \right) T(s) + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{1}{K_1} \right) N(s). \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_5(\alpha_2(s))$ is defined as

$$\begin{aligned} \varphi(x, h) = & \frac{1-h}{1+h} \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) - \\ & \beta + \frac{2h}{1+h} \left(\left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) + \right. \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right). \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_6(\alpha_2(s))$ is

$$\begin{aligned} \varphi(x, h) = & (1-h) \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) \cdot \beta \\ & + h \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right). \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_7(\alpha_2(s))$ is defined as

$$\begin{aligned} \varphi(x, h) = & e^h (1-h) \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right) - \beta + \\ & \frac{h}{2} \left(2h + \frac{1}{2} \right) \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + \left(C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2} \right) N(s) \left. \right). \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_8(\alpha_2(s))$ is defined by

$$\begin{aligned} \varphi(x, h) = & \frac{1-h}{1+h} \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + (C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) - \beta \right) + \frac{2h}{1+h} \left(\left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \right. \right. \\ & \left. \left. \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) \right). \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_9(\alpha_2(s))$ is defined as

$$\begin{aligned} \varphi(x, h) = & \ln e^{(1-h)} \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + (C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) - \beta \right) + \\ & \ln e^h \left((C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}) N(s) \right). \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_{10}(\alpha_2(s))$ is defined as

$$\begin{aligned} \varphi(x, h) = & e^h (1-h) \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + (C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) - \beta \right) + \frac{h}{2} \left(2h + \frac{1}{2} \right) \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \right. \right. \\ & \left. \left. \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right). \end{aligned}$$

The deformation retract of the helix $\alpha_2(s)$ into the retraction $r_{11}(\alpha_2(s))$ is defined as

$$\begin{aligned} \varphi(x, h) = & \ln e^{(1-h)} \left(\left(S - \frac{C_1 K_1}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_1}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1^2 S}{K_2^2 - K_1^2} \right) T(s) \right. \\ & + (C_1 \cos(S\sqrt{K_2^2 - K_1^2}) + C_2 \sin(S\sqrt{K_2^2 - K_1^2}) - \frac{K_1}{K_2^2 - K_1^2}) N(s) + \\ & \left. \left(\frac{-C_1 K_2}{\sqrt{K_2^2 - K_1^2}} \sin(S\sqrt{K_2^2 - K_1^2}) + \frac{C_2 K_2}{\sqrt{K_2^2 - K_1^2}} \cos(S\sqrt{K_2^2 - K_1^2}) + \frac{K_1 K_2 S}{K_2^2 - K_1^2} \right) B(s) - \beta \right) + \ln e^h \{0, 0, 0\}. \end{aligned}$$

Conclusion

Consider a curve in a space suppose that the curve is sufficiently smooth so that the Frenet frame adapted to is defined the curvature k_1 and torsion k_2 then provide a complete characterization of the curve .

Helix is one of the most fascinating curves in Science and Nature ,a helix is a geometry curve with non - vanishing constant curvature k_1 and non-vanishing constant torsion k_2 .A curve of constant slope or general helix in Euclidean 3-space E^3 ,is defined by the property that the tangent makes a constant angle with a fixed straight line .

In the present article, we obtain and study types of retraction of the position vectors of time like helix in Minkowski 3-space. Also ,by using the position vectors of the curve and retraction of the position vectors ,we deduced types of the deformation retracts of the time like helix in Minkowski 3-space. The relations between the retraction and the deformation retracts of time like helix are obtained. Types of minimal retraction of time like helix in Minkowski 3-space are also presented. New types of homotopy maps are deduced .

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