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Framelets in Quaternion Space

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Abstract: In this paper,, we will familiar with the quaternion space, extend the Fourier transform and classic wavelet transform, then introduce continuous quaternion wavelet and express admissible condition to quaternion wavelet transform. We also show that there can be established fundamental predicates such as inner multiplication, soft relationship and inversion formula when quaternion wavelet satisfies in particular admissible conditions.

Keywords: quaternion wavelet transform, admissible quaternion wavelet, wavelet transform, continuous quaternion wavelet, framelets.

Introduction

Researchers worked on quaternion algebra (1993) and extended transforming true Fourier to quaternion Fourier [1]. They found some predicates of Fourier transform in the space and could extend classical wavelet transform to quaternion algebra. They found some important predicates in this case. Hey and Zaw expressed continuous quaternion wavelet transform on quaternion functions and approved a number of the used predicates in classical Fourier transform on the extended wavelets [3]. Travershni has suggested discrete wavelet transform using quaternion Fourier transform that Birocrochano and Zaw stated its applications [2].
Recently, using Clifford Fourier transform kernel, there has been introduced an extension on wavelet transform to Clifford algebra[1]. In the present paper we are going to describe the desired theory compatible with new Fourier Kernel.

1.Quaternions and their Properties

William Hamilton (1843), a mathematician, discovered quaternions. After a long struggle, he could expand complex numbers into three-dimensional (3D) items. The necessity of extending operators on 3D vectors, including multiplication and division, caused Hamilton to bring up a four-dimensional algebra.
Field H on quaternion is a four-dimensional field on field R that k, j, i and 1 are its fundamental elements with following properties. It is true in Hamilton multiplication rule too.

$$i^2 = j^2 = k^2 = ijk = -1 \quad , \quad ij = -ji = k$$
$$jk = -kj = i \quad , \quad ki = -ik = j \tag{1}$$

The field identity member is 1. Each quaternion is defined as follows:

$$\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

We can consider the field of real numbers under H fields, so that (2)

$$\mathbb{R} = \{q \in \mathbb{H} \mid q = q_0 + i0 + j0 + k0\} \tag{3}$$

At least, there are three options for the field of complex numbers as follows that can be a sub-field on H:

$$\mathbb{C}_i = \{q \in \mathbb{H} \mid q = q_0 + iq_1 + j0 + k0\}$$

$$\mathbb{C}_j = \{q \in \mathbb{H} \mid q = q_0 + i0 + jq_2 + k0\} \qquad (4)$$

$$\mathbb{C}_k = \{q \in \mathbb{H} \mid q = q_0 + i0 + j0 + kq_3\}$$

According to 1, as H is non commutative, it cannot be expanded different results of combined figures to quaternion. For simplicity, we write a quaternion q as a scalar sum of q_0 in R and a vector $iq_1 + jq_2 + kq_3$ in R^3.

$$q = q_0 + q_v = q_0 + iq_1 + jq_2 + kq_3$$

That its scalar and mere parts are shown with $sc(q) = q_0$ and $V(q) = q_v$ respectively. Therefore, it can be written:

$$q = sc(q) + V(q) \qquad (6)$$

Conjugate of quaternion q is achieved by changing sign of its mere part, namely

$$\overline{q} = q_0 - q_v = q_0 - q_1 i - q_2 j - q_3 k \qquad (7)$$

Definition1.1 For an interval of $a \leq t \leq t$, space of $L^2([a,b])$ contains a set of all integrable functions on $a \leq t \leq t$. In other words:

$$L^2([a,b]) = \left\{ f \ : [a,b] \to c; \int_a^b |f(t)|^2 dt < \infty \right\}. \qquad (8)$$

Discontinuous functions may be a part of this space.
Inner product of L^2 is defined as follows:

$$\langle f, g \rangle_{L^2} = \int_a^b f(t)\overline{g(t)}dt \qquad f, g \in L^2([a,b])$$

In the similar way, $L^2(R^2, H)$ is defined with corresponding inner product.

2. Quaternion Fourier Transform (QFT)

Naturally, we can expand the Fourier transform to quaternion algebra. This expansion is generally called Quaternion Fourier Transform (QFT). Here, we examine some non commutative statements on quaternion. In the manner we have considered, there are three different types of Fourier quaternion transform:

1. Fourier quaternion transform of the left side

$$\mathcal{F}_q\{f\}(\omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} f(x) d^2x,$$

2. Fourier quaternion transform of the right side

$$\mathcal{F}_q\{f\}(\omega) = \int_{\mathbb{R}^2} f(x) e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} d^2x,$$

3. Dual Fourier quaternion transform

$$\mathcal{F}_q\{f\}(\omega) = \int_{\mathbb{R}^2} e^{-i\omega_1 x_1} f(x) e^{-j\omega_2 x_2} d^2x,$$

Here, we use QFT of the right side and explain its definitions and propositions.

Definition 2.1 : Quaternion Fourier transform on $f \in L^1(\mathbb{R}^2, \mathbb{H})$ is a function $\mathcal{F}_q\{f\} : \mathbb{R}^2 \longrightarrow \mathbb{H}$ defined as follows:

$$\mathcal{F}_q\{f\}(\omega) = \int_{\mathbb{R}^2} f(x) e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} d^2x$$

Where $x = x_1 e_1 + x_2 e_2$, $\omega = \omega_1 e_1 + \omega_2 e_2$ and nominal multiplication $e^{-i\omega_1 x_1} e^{-j\omega_2 x_2}$ are the kernel of quaternion Fourier.

Inverse of quaternion Fourier is defined as below:

$$f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\omega)\, e^{j\omega_2 x_2} e^{i\omega_1 x_1} d^2\omega$$

Remark : Except the $\frac{1}{(2\pi)^2}$, there are two general contract in inversion Fourier transform. One is achieved by substituting $\omega \longrightarrow 2\pi\omega$ in $\mathcal{F}_q = \frac{1}{2\pi} \int \ldots d^2x$ and $\mathcal{F}_q = \frac{1}{2\pi} \int \ldots d^2\omega$. All calculation of this article can be easily changed to other contracts.

If we use Euler's formula for quaternion Fourier kernel, then we can rewrite it as follows.

$$\begin{aligned}
\mathcal{F}_q\{f\}(\omega) &= \int_{\mathbb{R}^2} f(x) e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} d^2x \\
&= \int_{\mathbb{R}^2} f(x) \Big(\cos(\omega_1 x_1) - i \sin(\omega_1 x_1) \Big) . \Big(\cos(\omega_2 x_2) - j \sin(\omega_2 x_2) \Big) d^2x \\
&= \int_{\mathbb{R}^2} f(x) \cos(\omega_1 x_1) \cos(\omega_2 x_2) d^2x - \int_{\mathbb{R}^2} f(x)\, i \sin(\omega_1 x_1) \cos(\omega_2 x_2) d^2x \\
&- \int_{\mathbb{R}^2} f(x)\, j \cos(\omega_1 x_1) \sin(\omega_2 x_2) d^2x + \int_{\mathbb{R}^2} f(x)\, k \sin(\omega_1 x_1) \sin(\omega_2 x_2) d^2x
\end{aligned}$$

Proposition 2.2: Suppose that $f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$, then quaternion Fourier transform is invertible and its inverse is as follows.

$$\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}](x) = f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\omega)\, e^{j\omega_2 x_2} e^{i\omega_1 x_1} d^2\omega$$

Proof is based on the definition on inverse Fourier transform

$$\begin{aligned}
\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}](x) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_q f(x)\, e^{j\omega_2 x_2} e^{i\omega_1 x_1} d^2\omega \\
&= f(x)
\end{aligned}$$

3. Quaternion wavelets

Quaternion wavelet and quaternion Fourier transform could expand transforming through wavelet to quaternion wavelet. In this section, we determine admissible condition for quaternion Fourier transform as well as continuous quaternion wavelet transform to admissible quaternion wavelet(AQW).

Definition 3.1: Suppose that function $\psi \epsilon L^2(\mathbb{R}^2;\mathbb{H})$ satisfies in the following conditions:

$$c_\psi = \int_{so(2)} \int_{\mathbb{R}^+} |\widehat{\psi}(ar\_\theta(\omega))|^2 \frac{dad\theta}{a} \qquad (9)$$

Where c_ψ is real positive constant independent from ω that is true in $|\omega| = 1$, "a" is dilation coefficient and r_θ denoted the rotation with angle θ. Then we say that ψ is an admissible quaternion wavelet(AQW).

Lemma 3.2 If $\omega = |\omega|\omega_0$ و $\psi \epsilon AQW$ that $|\omega_0| = 1$, then (9) is real positive constant independent for $\omega \epsilon \mathbb{R}^2$.

Proof: as r_θ is linear and $\frac{da}{a}$ is Haar measure of multiplication group R+, then we have:

$$\begin{aligned} \int_{so(2)} \int_{\mathbb{R}^+} |\widehat{\psi}(ar_{-\theta}(\omega))|^2 \frac{dad\theta}{a} &= \int_{so(2)} \int_{\mathbb{R}^+} |\widehat{\psi}(ar_{-\theta}(|\omega|\omega_0))|^2 \frac{dad\theta}{a} \\ &= \int_{so(2)} \int_{\mathbb{R}^+} |\widehat{\psi}(a|\omega|r_{-\theta}(\omega_0))|^2 \frac{dad\theta}{a} \qquad (10) \\ &= \int_{so(2)} \int_{\mathbb{R}^+} |\widehat{\psi}(ar_{-\theta}(\omega_0))|^2 \frac{dad\theta}{a} \end{aligned}$$

■

Note that $\psi \epsilon AQW$ can be written as the following form:

$$\psi(x) = \psi_0(x) - \psi_1(x) - \psi_2(x) - \psi_3(x) \qquad (11)$$

,

where $\psi_s \epsilon L^2(\mathbb{R}^2;\mathbb{R})$ that $s = 0,1,2,3$.

Note: Using QFT definition and linearity specification of quaternion Fourier transform, we have

$$\begin{aligned} \mathcal{F}_q\{\psi\}(\omega) &= \int_{\mathbb{R}^2} (\psi_0(x) + i\psi_1(x) + j\psi_2(x) + k\psi_3(x))e^{-i\omega_1 x_1} e^{-j\omega_2 x_2} \qquad (12) \\ &= \mathcal{F}_q\{\psi_0\}(\omega) + i\mathcal{F}_q\{\psi_1\}(\omega) + j\mathcal{F}_q\{\psi_2\}(\omega) + k\mathcal{F}_q\{\psi_3\}(\omega) \end{aligned}$$

That we suppose $\mathcal{F}_q\{\psi_s\} \epsilon L^2(\mathbb{R}^2;\mathbb{R})$ for $s = 0,1,2,3$

Proposition 3.3 Suppose that ψ is admissible quaternion function. Quaternion Fourier Transform can be written as

$$\begin{aligned} \mathcal{F}_q\{\psi_{a,\theta,b}\}(\omega) = \; & e^{-i\omega_1 b_1}\{\widehat{\psi_0}(ar_{-\theta}(\omega) + i\widehat{\psi_1}(ar_{-\theta}(\omega))\}e^{-j\omega_2 b_2} \\ + \; & ae^{i\omega_1 b_1}\{j\widehat{\psi_2}(ar_{-\theta}(\omega) + k\widehat{\psi_3}(ar_{-\theta}(\omega))\}e^{-j\omega_2 b_2} \end{aligned} \qquad (13)$$

Proof: using QFT definition, we have

$$\mathcal{F}_q\{\psi_{a,\theta,b}\}(\omega) = \int_{\mathbb{R}^2} \frac{1}{a}\psi(r_{-\theta}(\frac{x-b}{a}))e^{-i\omega_1 x_1}e^{-j\omega_2 x_2}d^2x$$

In the above equilibrium, we perform variable of $\frac{x-b}{a} = y$ that we achieve:

$$\begin{aligned}\mathcal{F}_q\{\psi_{a,\theta,b}\}(\omega) &= \int_{\mathbb{R}^2} \frac{1}{a}\psi(r_{-\theta}(y))e^{-i\omega_1(b_1+ay_2)}e^{-j\omega_2(b_2+ay_2)}a^2d^2y \\ &= a\int_{\mathbb{R}^2} \psi(r_{-\theta}(y))e^{-i\omega_1 b_1}e^{-ia\omega_1 y_1}e^{-ja\omega_2 y_2}d^2y e^{-j\omega_2 b_2}\end{aligned}$$

Firstly, we observed that $\psi = \psi_0 + i\psi_1 + j\psi_2 + k\psi_3$. Also

$$\begin{aligned}j\psi_2 e^{-i\omega_1 b_1} &= j\psi_2(\cos(\omega_1 b_1) - i\sin(\omega_1 b_1)) \\ &= j\psi_2\cos(\omega_1 b_1) - ji\psi_2\sin(\omega_1 b_1)) \\ &= j\psi_2\cos(\omega_1 b_1) + k\psi_2\sin(\omega_1 b_1) \\ &= j\psi_2\cos(\omega_1 b_1) + ij\psi_2\sin(\omega_1 b_1) \\ &= j\psi_2(\cos(\omega_1 b_1) + i\sin(\omega_1 b_1)) \\ &= j\psi_2 e^{i\omega_1 b_1} = \overline{j\psi_2 e^{-i\omega_1 b1}}\end{aligned}$$

Therefore,

$$\begin{aligned}j\psi_2 e^{-i\omega_1 b_1} &= \overline{j\psi_2 e^{-i\omega_1 b1}} \\ &= \overline{\psi_2 e^{-i\omega_1 b1}}j \\ &= \psi_2 e^{i\omega_1 b_1}j\end{aligned}$$

Accordingly, we have:

$$k\psi_3 e^{-i\omega_1 b_1} = \psi_3 e^{i\omega_1 b_1}k$$

The above equilibrium shows

$$\begin{aligned}
\mathcal{F}_q\{\psi_{a,\theta,b}\}(\omega) &= a\int_{\mathbb{R}^2}\{\psi_0(r_{-\theta}(y))+i\psi_1(r_{-\theta}(y))+j\psi_2(r_{-\theta}(y))\\
&+ k\psi_3(r_{-\theta}(y))\}\times e^{-i\omega_1 b_1}e^{-ia\omega_1 y_1}e^{-ja\omega_2 y_2}d^2ye^{-j\omega_2 b_2}\\
&= a\int_{\mathbb{R}^2}(e^{-i\omega_1 b_1}\{(\psi_0(r_{-\theta}(y))+i\psi_1(r_{-\theta}(y))\}\\
&+ e^{i\omega_1 b_1}\{j\psi_2(r_{-\theta}(y))+k\psi_3(r_{-\theta}(y))\}\})\times e^{-i\omega_1 b_1}e^{-ia\omega_1 y_1}e^{-ja\omega_2 y_2}d^2ye^{-j\omega_2 b_2}\\
&= a\int_{\mathbb{R}^2}e^{-i\omega_1 b_1}\{\psi_0(r_{-\theta}(y))+i\psi_1(r_{-\theta}(y))\}\times e^{-i\omega_1 b_1}e^{-ia\omega_1 y_1}e^{-ja\omega_2 y_2}d^2ye^{-j\omega_2 b_2}\\
&+ a\int_{\mathbb{R}^2}e^{i\omega_1 b_1}\{j\psi_2(r_{-\theta}(y))+k\psi_3(r_{-\theta}(y))\}\times e^{-i\omega_1 b_1}e^{-ia\omega_1 y_1}e^{-ja\omega_2 y_2}d^2ye^{-j\omega_2 b_2}
\end{aligned},$$

so

$$\begin{aligned}
\mathcal{F}_q\{\psi_{a,\theta,b}\}(\omega) &= ae^{-i\omega_1 b_1}\int_{\mathbb{R}^2}\psi_0(r_{-\theta}(y))\times e^{-i\omega_1 b_1}e^{-ia\omega_1 y_1}e^{-ja\omega_2 y_2}d^2ye^{-j\omega_2 b_2}\\
&+ ae^{-i\omega_1 b_1}\int_{\mathbb{R}^2}i\psi_1(r_{-\theta}(y))\times e^{-i\omega_1 b_1}e^{-ia\omega_1 y_1}e^{-ja\omega_2 y_2}d^2ye^{-j\omega_2 b_2}\\
&+ ae^{i\omega_1 b_1}\int_{\mathbb{R}^2}j\psi_2(r_{-\theta}(y))\times e^{-i\omega_1 b_1}e^{-ia\omega_1 y_1}e^{-ja\omega_2 y_2}d^2ye^{-j\omega_2 b_2}\\
&+ ae^{i\omega_1 b_1}\int_{\mathbb{R}^2}k\psi_3(r_{-\theta}(y))\times e^{-i\omega_1 b_1}e^{-ia\omega_1 y_1}e^{-ja\omega_2 y_2}d^2ye^{-j\omega_2 b_2}\\
&= ae^{-i\omega_1 b_1}\widehat{\psi_{0l}}(ar_{-\theta}(\omega))e^{-j\omega_2 b_2}+ae^{i\omega_1 b_1}\widehat{\psi_{1l}}(ar_{-\theta}(\omega))e^{-j\omega_2 b_2}
\end{aligned}.$$

That shows

$$\begin{aligned}
\widehat{\psi_{0l}}(ar_{-\theta}(\omega)) &= \widehat{\psi_0}(ar_{-\theta}(\omega))+i\widehat{\psi_1}(ar_{-\theta}(\omega))\\
\widehat{\psi_{1l}}(ar_{-\theta}(\omega)) &= j\widehat{\psi_2}(ar_{-\theta}(\omega))+k\widehat{\psi_3}(ar_{-\theta}(\omega))
\end{aligned} \qquad (14)$$

Accordingly, it will be approved completely.

4. Continuous Quaternion Wavelet Transform(CQWT)

We define Continuous Quaternion Wavelet Transform (CQWT) on a quaternion function $f\epsilon L^2(\mathbb{R}^2:\mathbb{H})$ in compared to $\psi\epsilon AQW$.

$$T_\psi : L^2(\mathbb{R}^2:\mathbb{H}) \longrightarrow L^2(\mathbb{R}^2:\mathbb{H})$$

$$\begin{aligned}
f \longmapsto T_\psi f(a,\theta,b) &= \langle f,\psi_{a,\theta,b}\rangle_{L^2(\mathbb{R}^2:\mathbb{H})}\\
&= \int_{\mathbb{R}^2}\frac{1}{a}\overline{\psi(r_{-\theta}(\frac{x-b}{a}))}d^2x
\end{aligned}$$

It is necessary to mention that arrangement statements in predicate of "suppose that $\psi\epsilon L^2(\mathbb{R}^2;\mathbb{H})$ is a quaternion wavelet" cannot be moved. In fact, we have this lemma.

Lemma 4.1 Suppose $\psi\epsilon AQW$. If $\psi\epsilon L^2(\mathbb{R}^2:\mathbb{H})$, then $CQWT$ is as follows:

$$T_\psi f(a,\theta,b) = \frac{a}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{f}(\omega)e^{-j\omega_2 b_2}\{\overline{\widehat{\psi_{ol}}(ar_{-\theta}(\omega))e^{i\omega_1 b_1}} + \overline{\widehat{\psi_{1l}}(ar_{-\theta}(\omega))e^{-i\omega_1 b_1}}\}d^2\omega \quad (16)$$

That $\widehat{\psi_{ol}}(ar_{-\theta}(\omega)), \widehat{\psi_{1l}}(ar_{-\theta}(\omega))$ have been defined in predicate of "suppose $\psi \epsilon L^2(\mathbb{R}^2;\mathbb{H})$ is a quaternion wavelet.

Proof

$$\begin{aligned}
T_\psi f(a,\theta,b) &= \langle f, \psi_{a,\theta,b}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} \\
&= \frac{1}{(2\pi)^2}\langle \widehat{f}, \widehat{\psi_{a,\theta,b}}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} a\widehat{f}(\omega)\overline{\widehat{\psi_{a,\theta,b}}(\omega)}d^2\omega \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} a\widehat{f}(\omega)\overline{(e^{-i\omega_1 b_1}\widehat{\psi_{0l}}(ar_{-\theta}(\omega))e^{-j\omega_2 b_2} + e^{i\omega_1 b_1}\widehat{\psi_{1l}}(ar_{-\theta}(\omega))e^{-j\omega_2 b_2})}d^2\omega \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} a\widehat{f}(\omega)\overline{(e^{-i\omega_1 b_1}\widehat{\psi_{0l}}(ar_{-\theta}(\omega))e^{-j\omega_2 b_2})}d^2\omega \\
&+ \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} a\widehat{f}(\omega)\overline{(e^{i\omega_1 b_1}\widehat{\psi_{1l}}(ar_{-\theta}(\omega))e^{-j\omega_2 b_2})}d^2\omega \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} a\widehat{f}(\omega)e^{j\omega_2 b_2}\overline{\widehat{\psi_{0l}}(ar_{-\theta}(\omega))}e^{i\omega_1 b_1}d^2\omega \\
&+ \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} a\widehat{f}(\omega)e^{j\omega_2 b_2}\overline{\widehat{\psi_{1l}}(ar_{-\theta}(\omega))}e^{-i\omega_1 b_1}d^2\omega
\end{aligned}$$

5. Reconstruction Formula

In this section, we show that we can achieve quaternion function f by continuous quaternion wavelet transform that the quaternion wavelet is true in the following condition.

Proposition 5.1 Suppose that $\psi = \psi_0 + i\psi_1 + j\psi_2 + k\psi_3 \epsilon AQW$ is an admissible continuous quaternion wavelet. Suppose that $\mathcal{F}_q\{\psi\} = \mathcal{F}_q\{\psi_0\}\epsilon\mathbb{R}$. Then for each $f, g\epsilon L^2(\mathbb{R}^2;\mathbb{H}) \cap L^1(\mathbb{R}^2;\mathbb{H})$ we have

$$\langle T_\psi f, T_\psi g\rangle_{L^2(\varrho;\mathbb{H})} = c_\psi\langle f, g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} \quad (17)$$

Proof: we use Plancherel Formula for transforming quaternion Fourier:

$$\begin{aligned}
\langle T_\psi f, T_\psi g\rangle_{L^2(\varrho;\mathbb{H})} &= \int_{so(2)}\int_{\mathbb{R}+}(\int_{\mathbb{R}^2} T_\psi f(a,\theta,b)\overline{T_\psi g(a,\theta,b)}d^2b)d\mu \\
&= \frac{1}{(2\pi)^2}\int_{so(2)}\int_{\mathbb{R}+}(\int_{\mathbb{R}^2}\mathcal{F}_q(T_\psi f(a,\theta,b))\mathcal{F}_q(\overline{T_\psi g(a,\theta,b)}) \\
&= \frac{1}{(2\pi)^2}\int_{so(2)}\int_{\mathbb{R}+}(\int_{\mathbb{R}^2}\mathcal{F}_q(T_\psi f(a,\theta,b))\overline{\mathcal{F}_q}(T_\psi g(a,\theta,b))d^2\omega)d\mu
\end{aligned}$$

According predicate of "suppose that $\psi \epsilon L^2(\mathbb{R}^2;\mathbb{H})$ is a quaternion wavelet", we have:

$$\begin{aligned}\langle T_\psi f, T_\psi g\rangle_{L^2(\varrho;\mathbb{H})} &= \frac{1}{(2\pi)^2}\int_{so(2)}\int_{\mathbb{R}+}(\int_{\mathbb{R}^2} a^2\widehat{f}(\omega)\widehat{\psi}(ar_{-\theta}(\omega))\overline{\widehat{g}(\omega)\widehat{\psi}(ar_{-\theta}(\omega))}d^2\omega)d\mu \\ &= \frac{1}{(2\pi)^2}\int_{so(2)}\int_{\mathbb{R}+}(\int_{\mathbb{R}^2} a^2\widehat{f}(\omega)\widehat{\psi}(ar_{-\theta}(\omega))\overline{\widehat{\psi}(ar_{-\theta}(\omega))}\overline{\widehat{g}(\omega)}d^2\omega)d\mu\end{aligned}$$

We change integral arrangement using Fubini theorem. We know that

$$C_\psi = \int_{so(2)}\int_{\mathbb{R}+}|\widehat{\psi}(ar_{-\theta}(\omega))|^2\frac{dad\theta}{a}, \qquad d\mu(a,\theta) = \frac{dad\theta}{a^3}$$

That is a real positive constant. As a result:

$$\begin{aligned}\langle T_\psi f, T_\psi g\rangle_{L^2(\varrho;\mathbb{H})} &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{f}(\omega)(\int_{so(2)}\int_{\mathbb{R}+}|\widehat{\psi}(ar_{-\theta}(\omega))|^2\frac{dad\theta}{a})\overline{\widehat{g}(\omega)}d^2\omega) \\ &= C_\psi\int_{\mathbb{R}^2}\widehat{f}(\omega)\overline{\widehat{g}(\omega)}d^2\omega \\ &= C_\psi\int_{\mathbb{R}^2} f(x)\overline{g(x)}d^2x = C_\psi\langle f,g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\end{aligned}$$

□

Proposition 5.2 (inversion formula):As hypotheses of Proposition 5.1, we can decompose any quaternion function $f\epsilon L^2(\mathbb{R}^2;\mathbb{H})$, as follows:

$$f(x) = \frac{1}{C_\psi}\int_\varrho T_\psi f(a,\theta,b)\psi_{a,\theta,b}d\lambda \qquad (18)$$

That is an integral in weak convergence.

Proof: we use proposition 5.1 for each $g\epsilon L^2(\mathbb{R}^2;\mathbb{H})$

$$\begin{aligned}C_\psi\langle f,g\rangle L^2(\mathbb{R}^2;\mathbb{H}) &= \int_{so(2)}\int_{\mathbb{R}+}(\int_{\mathbb{R}^2} T_\psi f(a,\theta,b)\overline{T_\psi g(a,\theta,b)}d^2b)d\mu \\ &= \int_\varrho T_\psi f(a,\theta,b)\overline{T_\psi g(a,\theta,b)}d\lambda \\ &= \int_\varrho T_\psi f(a,\theta,b)\overline{(\int_{\mathbb{R}^2} g(x)\overline{\psi_{a,\theta,b}(x)}d^2x)}d\lambda \\ &= \int_\varrho\int_{\mathbb{R}^2} T_\psi f(a,\theta,b)\psi_{a,\theta,b}(x)\overline{g(x)}d^2x \\ &= \int_{\mathbb{R}^2}\int_\varrho T_\psi f(a,\theta,b)\psi_{a,\theta,b}(x)\overline{g(x)}d\lambda d^2x) \\ &= \langle\int_\varrho T_\psi f(a,\theta,b)\psi_{a,\theta,b}(x)d\lambda, g(x)\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}\end{aligned}$$

As there is identity feature of inner product for each $g\epsilon L^2(\mathbb{R}^2;\mathbb{H})$, we conclude that

$$C_\psi f(x) = \int_\varrho T_\psi f(a,\theta,b)\psi_{a,\theta,b}(x)d\lambda \qquad (19)$$

This completes the proof. ■

Proposition 5.3 (kernel reconstruction): Suppose that $\psi \epsilon AQW$. If $K_\psi(a,\theta,b;a',\theta',b') = C_\psi^{-1}\langle\psi_{a,\theta,b};\psi_{a',\theta',b'}\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}$, then $K_\psi(a,\theta,b;a',\theta',b')$ is a kernel reconstruction in , $L^2(\varrho,d\lambda)$ namely:

$$T_\psi f(a',\theta',b') = \int_\rho T_\psi f(a,\theta,b)K_\psi(a,\theta,b;a',\theta',b')d\lambda$$

Proof: by substituting (18) in CQWT definition, we have:

$$\begin{aligned} T_\psi f(a',\theta',b') &= \int_{\mathbb{R}^2} f(x)\overline{\psi_{a',\theta',b'(x)}}d^2x \\ &= \int_{\mathbb{R}^2}(C_\psi^{-1}\int_\varrho T_\psi f(a,\theta,b)\psi_{a,\theta,b}(x)d\lambda)\overline{\psi_{a',\theta',b'}(x)}d^2x \\ &= \int_\varrho T_\psi f(a,\theta,b)(C_\psi^{-1}\int_{\mathbb{R}^2}\psi_{a,\theta,b}(x))\overline{\psi_{a',\theta',b'}(x)}d^2x)d\lambda \\ &= \int_\varrho T_\psi f(a,\theta,b)K_\psi(a,\theta,b;a',\theta',b')d\lambda \end{aligned}$$

And we re done.

Reference:

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